Problem Set 1

Feb 20, 2025

In this problem set, we always assume that the groups have finite order and the vector spaces are finite dimensional over $\mathbb{C}$.

Problem 1. (Lecture 2 exercise) Let V be a vector space, and $p : V \to V$ a projection, that is, a linear map satisfying $p^2 = p$. Show that $V = \mathrm{Ker}(p) \oplus \mathrm{Im}(p)$.

Problem 2. (Lecture 3 exercise) Let (π_1, V) and (π_2, W) be two representations of G. Let (π_1^*, V^*) the dual representation of (π_1, V). We further assume that $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V^* = n$ and $\dim_{\mathbb{C}} W = m$. Let $\mathrm{Hom}_{\mathbb{C}}(V, W)$ be the vector space of linear maps $T : V \to W$.

1. Let $\{f_1, \ldots, f_n\}$ be a basis for V^* and $\{w_1, \ldots, w_m\}$ a basis for W. Then we can construct a linear map
$$\begin{aligned} \mathcal{A} : \qquad & V^* \otimes W \to \mathrm{Hom}_{\mathbb{C}}(V, W) \\ & \sum_{i,j} \lambda_{i,j} f_i \otimes w_j \mapsto \left(v \mapsto \sum_{i,j} \lambda_{i,j} f_i(v) w_j \right) \end{aligned}$$
Show that $\mathcal{A}$ is a (vector space) isomorphism between $V^* \otimes W$ and $\mathrm{Hom}_{\mathbb{C}}(V, W)$.
2. In Lecture 3, we defined a representation $(\widetilde{\pi}, \mathrm{Hom}_{\mathbb{C}}(V, W))$. Show that $\mathcal{A} : (\pi_1^* \otimes \pi_2, V^* \otimes W) \to (\widetilde{\pi}, \mathrm{Hom}_{\mathbb{C}}(V, W))$ is a (representation) isomorphism.

Problem 3. Let (π_1, V) be an 1-dimensional representation of G and (π_2, W) an irreducible representation of G. Show that $(\pi_1 \otimes \pi_2, V \otimes W)$ is an irreducible representation of G.

Problem 4. (Lecture 6 exercise) Let G be a group and denote by $C(G)$ its (in-equivalent) conjugate classes. For $[g] \in C(G)$, we can define a function
$$f_{[g]}(h) = \begin{cases} 1 & \text{if } h \in [g] \\ 0 & \text{if } h \notin [g] \end{cases}$$
Let H be the vector space of class functions in Lecture 6. Show that $\{f_{[g]} : [g] \in C(G)\}$ is a basis for H.

Problem 5. Denote by K the Kelin four group $K =< a, b | a^2 = b^2 = (ab)^2 = e >$. Show that K is an abelian group of order 4. Then calculate the character table for K.