Problem Set 2

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In this problem set, we always assume that s is a real variable, and the notation p represents primes numbers.

Problem 1. (Lecture 10 exercise) Denote by φ the Euler function defined by $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$ for $n \geq 1$.

1. Show that φ is a multiplicative function and $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.
2. We define the series $L(s) = \sum_{n\geq 1} \frac{\varphi(n)}{n^s}$. Show that $L(s)$ is absolutely convergent when $s > 2$ and $L(s) = \frac{\zeta(s-1)}{\zeta(s)}$. Here $\zeta(s)$ is the Riemann zeta function.

Hint: φ is multiplicative $\Rightarrow L(s) = \prod_p \left(1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + \frac{\varphi(p^3)}{p^{3s}} + \cdots\right)$, provided $s > 2$. Then show $\left(1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + \frac{\varphi(p^3)}{p^{3s}} + \cdots\right) = \frac{\left(1-\frac{1}{p^s}\right)}{\left(1-\frac{1}{p^{s-1}}\right)}$.

Problem 2. (Lecture 12 exercise) Let $q \geq 1$ be an integer. Show that, for any $n \in \mathbb{Z}$,

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod q \\ 0 & \text{otherwise} \end{cases}$$

Problem 3. Let $\zeta(s)$ be the Riemann zeta function. Show that

$$\lim_{s\to 1^+} (s-1)\zeta(s) = 1.$$

Hint: in Lecture 9, we found the upper bound and the lower bound for $\zeta(s)$. Apply the Squeezing Theorem.