Problem Set 3

Mar 13, 2025

Problem 1. (Lecture 14 exercise) Let R be a commutative ring and M an R-module. Show that the following conditions are equivalent:

1. M is a Noetherian R-module.
2. Every ascending chain of submodules will eventually become constant; that is, if we have a chain of submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \cdots$, then we can find some $n \geq 1$ such that $M_n = M_{n+1} = M_{n+2} = \cdots$.
3. Every nonempty set S of submodules will contain a maximal element, that is, we can find a submodule $M_0 \in S$ and M_0 is not properly contained in any submodule in S.

Hint: look at the equivalent conditions for a Noetherian ring.

Problem 2. (Lecture 16 exercise) Let V be a finite-dimensional vector space over a number field F. Let $B : V \times V \to \mathbb{C}$ be a nondegenerate bilinear form. Let $\{v_1, \ldots, v_n\}$ be a basis for V.

1. Show that the map
$$\begin{aligned} \Phi : & V \to V^* \\ & v \mapsto B_v \end{aligned}$$
is a (vector space) isomorphism between V and V^*.
2. There exists another basis $\{w_1, \ldots, w_n\}$ such that
$$B(w_j, v_i) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Problem 3. Let $\alpha \in \mathbb{C}$ be a complex number. We say that α is an *algebraic number* if we can find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.

1. Show that α is an algebraic number if and only if $\mathbb{Q}[\alpha]$ is a finite-dimensional $\mathbb{Q}$-vector space.
2. Let $\overline{\mathbb{Q}}$ be the set of all algebraic numbers. Show that $\overline{\mathbb{Q}}$ is a field, and $\overline{\mathbb{Q}}$ is the union of all number fields.

Hint: consider how we showed the ring of integers is a ring. Additionally, you can directly use the fact that $\mathbb{Q}[\alpha]$ is a field provided that α is an algebraic number.