Some Notations:

Let R be a commutative ring with multiplicative identity 1. $(R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, F)$ a field

For an integer $n \geq 1$, we define: (general linear group)

$GL_n(R) = \{$ invertible $n \times n$ matrices with all entries in $R \}$

Fact: A matrix $A \in GL_n(R)$ is invertible if and only if $\det(A)$ is invertible in R.

This implies:

$GL_n(\mathbb{Z}) = \{A :$ $n \times n$ matrices with all entries in $\mathbb{Z}$ and $\det(A) = \pm 1 \}$

Let F be a field. $(F = \mathbb{Q}, \mathbb{R}, \mathbb{C})$

$GL_n(F) = \{A :$ $n \times n$ matices with all entries in F and $\det(F) \neq 0 \}$

Let F be a field and V a vector space over F.

We assume that $\dim_F V = n < \infty$.

Then we can find a basis $\{v_1, \cdots v_n\} \subseteq F$

such that

(1) $V = \operatorname{span}_F \{v_1, v_2, \cdots v_n\}$

(2) $v_1, \cdots v_n$ are linearly independent.

We call $\{v_1, \cdots v_n\}$ is a <u>basis</u> for V.

Remark: We have ∞-many choices of basis.

However, the number of elements in a basis is always n. (That is how we define the dimension of a vector space.)

<u>Definition</u>: Let V be a vector space. An <u>isomorphism</u> is a bijective linear map $T: V \to V$.

Then we set:

$$GL(V) = \{ \text{isomorphisms of a vector space } V \}.$$

Lemma: Assume that $\dim_F V = n$. Then there is a bijection between $GL(V)$ and $GL_n(F)$.

Remark: This lemma implies, all the isomorphisms between V and V are indeed invertible matrices.

Complex numbers.

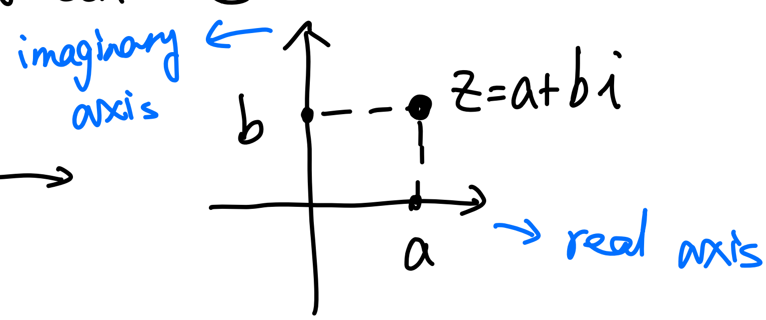
Recall: $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, \ i^2 = -1\}$

For $z = a + bi$,

- a is called the real part of z, denoted by $\mathrm{Re}(z)$
- b is called the imaginary part of z, denoted by $\mathrm{Im}(z)$.

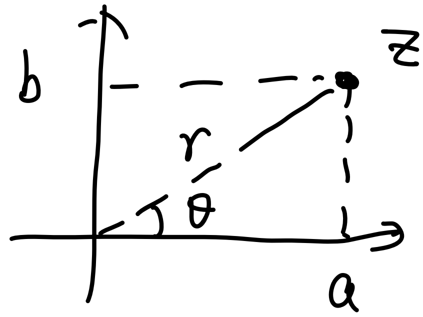
There is an bijection between $\mathbb{C}$ and $\mathbb{R}^2$.

$z = a + bi \longmapsto$



This also implies: we can view $\mathbb{C}$ as a $\mathbb{R}$-vector space and $\dim_{\mathbb{R}} \mathbb{C} = 2$.

By Calculus II, we can express a point in $\mathbb{R}^2$ by its polar coordinates:



$$z = a + b \cdot i$$

$$= r\cos\theta + r\sin\theta \cdot i$$

Euler's identity

$$= r(\cos\theta + i\sin\theta) \Leftarrow e^{i\theta} = \cos\theta + i\sin\theta.$$

$$= re^{i\theta}$$

with $r \geq 0$ and $\theta \in \mathbb{R}$.

Note: if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

Definition: Let $z = a + bi$. Then the conjugate of z, denoted by $\bar{z}$, is defined by:

$$\bar{z} = a - bi$$

Check: $z \cdot \bar{z} = a^2 + b^2$.

and we set $|z| = (z \cdot \bar{z})^{\frac{1}{2}} = (a^2 + b^2)^{\frac{1}{2}}$.

Next, we set

$$\mathbb{C}[x] = \{\text{polynomials with coefficients in } \mathbb{C}\}.$$

Theorem (Fundamental theorem of algebra)

Let $f(x) \in \mathbb{C}[x]$ and $f(x)$ is not a constant.

Then $f(x)$ has a root in $\mathbb{C}$.

Corollary: Let $f(x) \in \mathbb{C}[x]$ of degree n.

$$f(x) = a_0 (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_n).$$

with $a_0, \alpha_1, \cdots \alpha_n \in \mathbb{C}$.

Proof of corollary: By fundamental theorem of $f(x)$,

$f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$.

Notice that $\mathbb{C}$ is a field $\Rightarrow$ $\mathbb{C}[x]$ is a Euclidean domain. Then we can find $q(x)$ and $r(x)$ s.t.

$$f(x) = q(x)(x-\alpha_1) + r(x)$$ with

$$\deg r(x) < \deg(x-\alpha_1) = 1 \quad (r(x) \text{ is a constant})$$

Notice that $f(\alpha_1) = 0$, $q(\alpha_1)(\alpha_1-\alpha_1) = 0 \Rightarrow r(\alpha_1) = 0$.

$r(x)$ is a constant and $r(\alpha_1) = 0 \Rightarrow r(x) = 0$.

This shows: $f(x) = q(x)(x-\alpha_1)$.

Then we set $f_1(x) = \dfrac{f(x)}{x-\alpha_1} = q(x)$.

We apply the fundamental theorem again and we have $f_1(x) = q_1(x)(x-\alpha_2)$

We continue this process and we finally get:

$$f(x) = a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n). \qquad \square$$

Let G be a finite group.

Let V be a finite dimensional vector space over $\mathbb{C}$.

Definition: A (linear) <u>representation</u> of G is a group homomorphism: $\pi: G \to GL(V)$, that is,

(1) for any $g \in G$, $\pi(g) \in GL(V)$

(2) for any $g_1, g_2 \in G$, $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$

(3) $\pi(e) = Id_V$.

Here e is the identity of G

Id_V is the isomorphism on V sending v to v.

We use the notation (π, V) for a representation of G.

Furthermore, if $\dim_{\mathbb{C}} V = n$, we say that the degree / dimension of π is n.

We use the notation: $\deg \pi = \dim \pi = n$.

Remark: If $\dim_{\mathbb{C}} V = n$, we can identify $GL(V)$ with $GL_n(\mathbb{C})$. This means, when fixed a basis of V, $\pi(g)$ can be identified with an $n \times n$ invertible matrix of $GL_n(\mathbb{C})$.

Remark: Notice that $\pi(g) \in GL(V)$ and $\pi(g)$ is a linear map and hence $\pi(g)$ is totally determined by $\pi(g)v_1, \cdots \pi(g)v_n$ provided that $\{v_1, \cdots v_n\}$ is a basis.

Lemma: Let (π, V) be a representation of G. Then

$$\pi(g^{-1}) = \pi(g)^{-1}.$$

Proof: This is due to:

$$\left.\begin{aligned} 1_V = \pi(e) &= \pi(g \cdot g^{-1}) = \pi(g) \cdot \pi(g^{-1}) \\ &= \pi(g^{-1} \cdot g) = \pi(g^{-1}) \cdot \pi(g). \end{aligned}\right\} \Rightarrow \pi(g^{-1}) = \pi(g)^{-1}$$

$\square$

Example I: Let $V = \mathbb{C}$ ($\dim_{\mathbb{C}} V = 1$)

Then $GL(V) = \mathbb{C}^{\times}$ $\left(\begin{array}{ccc} \mathbb{C}^{\times} & \longrightarrow & GL(V) \\ a & \mapsto & (v \mapsto av) \end{array} \right)$

The <u>trivial representation</u> is:

$$\pi(g) = 1 \quad \text{for all } g \in G.$$

Remark: The trivial representation can be defined for arbitrary groups. However, in most cases, the representation is dependent on the group G.

Example II: Let $G = S_n = \{\text{permutations of } n \text{ numbers}\}$

standard basis.

$$V = \mathbb{C}^n = \mathrm{span}_{\mathbb{C}} \{e_1, \cdots e_n\}.$$

Then we define a representation (π, V)

$$\pi : \begin{array}{ccc} S_n & \longrightarrow & GL(V) \\ \sigma & \longmapsto & \pi(\sigma) \end{array}$$

with $\pi(\sigma)\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i e_{\sigma(i)}$

This is called the <u>standard representation</u> for S_n.

Another way to consider this representation is:

V has a standard basis $\{e_1, \cdots e_n\}$

Then $\pi(\sigma)$ is totally determined by $\pi(\sigma) e_i$

and we know: $\pi(\sigma) e_i = e_{\sigma(i)}$.

Sometimes we use the notation:

$(\mathrm{std}, \mathbb{C}^n)$ for the standard representation of S_n.

Example III: Group Algebra and right translation.

(Next class)