Definition: Let G be a finite group. We define the <u>group algebra</u>, denoted by, $\mathbb{C}[G]$ to be the set of formal sums: $\sum_{g\in G} a_g \cdot g$ with $a_g \in \mathbb{C}$.

$$\mathbb{C}[G] := \left\{ \sum_{g\in G} a_g \cdot g : a_g \in \mathbb{C}, \right\}$$

We can define the following operations:

(1) addition: $\sum_{g\in G} a_g \cdot g + \sum_{g\in G} b_g \cdot g = \sum_{g\in G} (a_g + b_g) \cdot g$.

$(\mathbb{C}[G], +)$ is an abelian group.

(2) scalar multiplication by $\mathbb{C}$: $\lambda \in \mathbb{C}$, $\sum_{g\in G} a_g\, g$

$$\lambda \cdot \sum_{g\in G} a_g \cdot g = \sum_{g\in G} (\lambda a_g) \cdot g$$

The scalar multiplication and the addition will make $\mathbb{C}[G]$ a vector space $/_{\mathbb{C}}$

and $\dim_{\mathbb{C}} \mathbb{C}[G] = |G|$ $\leftarrow$ the cardinality / order of G.

Remark: Sometimes we use notation "e_g" instead of g.

In this case $\{ e_g : g \in G \}$ is a basis for $\mathbb{C}[G]$.

(3) the multiplication:

$$\left(\sum_{s \in G} a_s \cdot s \right) \cdot \left(\sum_{t \in G} b_t \cdot t \right) = \sum_{s,t \in G} a_s b_t \cdot (st)$$

$$= \sum_{g \in G} \left(\sum_{st=g} a_s b_t \right) g$$

Example / Definition: Let G be a finite group with its group algebra $\mathbb{C}[G]$, the <u>regular representation</u> $(R, \mathbb{C}[G])$ is defined by:

$$R: G \longrightarrow GL(\mathbb{C}[G])$$

$$g \longmapsto R(g)$$

where $R(g)\left(\sum_{t \in G} a_t\, t \right) = \sum_{t \in G} a_t \cdot gt$

(or $R(g)\left(\sum_{t \in G} a_t\, e_t \right) = \sum_{t \in G} a_t\, e_{gt}$.

We only check the homomorphism part, that is

$$R(g_1 g_2) = R(g_1) R(g_2)$$

Pick an arbitrary $\sum_{t \in G} a_t \cdot t \in \mathbb{C}[G]$

$$R(g_1 g_2)\left(\sum_{t \in G} a_t \cdot t\right) = \sum_{t \in G} a_t \cdot g_1 g_2 t$$

$$\begin{aligned} R(g_1) R(g_2)\left(\sum_{t \in G} a_t t\right) &= R(g_1)\left(R(g_2) \sum_{t \in G} a_t t\right) \\ &= R(g_1)\left(\sum_{t \in G} a_t \cdot g_2 t\right) \\ &= \sum_{t \in G} a_t \cdot g_1 g_2 t = R(g_1 g_2)\left(\sum_{t \in G} a_t t\right). \end{aligned}$$

Sime we chose $\sum_{t \in G} a_t \cdot t$ arbitrarily, we showed:

$$R(g_1 g_2) = R(g_1) R(g_2)$$

Remark: This representation is very important in later chapters, since all the irreducible representations (the building blocks) will show up in the regular representation.

Subrepresentations.

Let (π, V) be a representation of G, and $W \leq V$ be a subspace.

Definition: We say that W is <u>stable</u> under the action of G if for any $g \in G$ and any $w \in W$,

$$\pi(g)\, w \in W.$$

In this case, we get a representation of G is the following way:

$$G \longrightarrow GL(W)$$
$$g \longmapsto \pi(g)$$

sime $\pi(g) w \in W$ for all $g \in G$ and $w \in W$.

Definition: We use (π^W, W) for this representation. and this is called a <u>subrepresentation</u> of (π, V).

Recall: In the linear algebra class, we have a the following result: let $W \subset V$ be two vector spaces, then we can a $W' \leq V$, a vector space, such that

$$V = W \oplus W'.$$

$\leftarrow$ complementary.

Our goal for this lecture is to prove a similar result:

Theorem: (Maschke) Let (π, V) be a repn of G and (π^W, W) be a subrepn of G. Then we can find $W_0 \subseteq V$, such that:

(1) W_0 is stable under G, i.e., (π^{W_0}, W_0) is a subrepn of (π, V)

(2) $V = W \oplus W_0$.

Remark: This theorem is true not only for vector spaces$/\mathbb{C}$. Indeed, if we assume $\mathrm{Char}(F) \nmid |G|$, then the theorem holds for vector spaces $/F$.

Before the proof, we need some preparations:

Definition: Let (π_1, V) and (π_2, W) be two repns of G. An <u>intertwining operator</u> is a linear map $T: V \to W$ such that, for any $g \in G$,

$$T \circ \pi_1(g) = \pi_2(g) \circ T.$$

or the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi_1(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\pi_2(g)"] \\ V \arrow[r, "T"'] & W \end{tikzcd}$$ for any $g \in G$.

We use $\mathrm{Hom}_G(V, W)$ for the set of all intertwining operators.

Definition: An intertwining operator $T: V \to W$ is an <u>isomorphism</u> if T is also a bijection between V and W.

In this case, we say that two repns (π_1, V) and (π_2, W) are isomorphic.

Definition: Let T be an intertwining operator for (π_1, V) and (π_2, W). Then

(1) $\ker(T) = \{ v \in V : T(v) = 0 \}$

(2) $\mathrm{Im}(T) = \{ w \in W : \text{there exists } v \in V \text{ s.t. } T(v) = w \}$

Lemma: $\ker(T)$, $\mathrm{Im}(T)$ are stable under G.

This means: $\ker(T)$ is a subrepn of V

$\mathrm{Im}(T)$ is a subrepn of W.

Proof: (1) Let $v \in \ker(T)$ and $g \in G$.

$$T(\pi_1(g)v) = \pi_2(g) \circ T(v) = 0 \Rightarrow \pi_1(g) v \in \ker(T)$$

(2) Let $w \in \mathrm{Im}(T)$ and $g \in G$

Then we can find $v \in V$ s.t. $w = T(v)$

$$\pi_2(g) w = \pi_2(g) \circ Tv = T(\pi_1(g) v) \in \mathrm{Im}(T).$$ ∎

Some linear algebra results:

Definition: Let V be a vector space. A <u>projection</u> is a linear map $p: V \to V$ satisfying $p \circ p = p$.

Exercise: Let V be a vector space and $p: V \to V$ a projection.

Then $V = \mathrm{Im}(p) \oplus \ker(p)$.

Hint: for any $v \in V$, $v = p(v) + (v - p(v))$.

show: $p(v) \in \mathrm{Im}(p)$, $v - p(v) \in \ker(p)$ and $\mathrm{Im}(p) \cap \ker(p) = \{0\}$

Observation: Let V be a vector space and suppose it can be written as a direct sum $V = W \oplus W'$. Then we can construct a projection in the following way:

$$V = W \oplus W' \qquad p: V \to V$$
$$v = w + w' \qquad v \mapsto w.$$

Notice that $P|_W = Id_W$ and $P|_{W'} = \{\vec{0}\}$

That is why we call p a projection.

Proof of Theorem: Suppose that (π^W, W) is a subrepn of (π, V). Then we can find $W' \leq V$, a subspace s.t.

$$V = W \oplus W'.$$

$\operatorname{Im} p = W$

$\ker p = W'$.

Then we can construct a projection $p: V \to V$.

Next, we construct a new map: $p_0: V \to V$

$$p_0 = \frac{1}{|G|} \sum_{g \in G} \pi(g) \circ p \circ \pi(g^{-1})$$

Claim: (1) p_0 is a projection

(2) $p_0: V \to V$ is an intertwining operator.

(3) $\operatorname{Im} p_0 = W$

We first assume the claims and finish the proof of the theorem:

Since p_0 is a projection, we can write

$$V = \operatorname{Im}(p_0) + \ker(p_0)$$

Since p_0 is an intertwining operator,

$\operatorname{Im}(p_0)$ and $\ker(p_0)$ are subrepns of G.

Since $\operatorname{Im}(p_0) = W$, we can set $W_0 = \ker(p_0)$

then $V = \operatorname{Im}(p_0) \oplus \ker(p_0) = W \oplus W_0$

and W_0 is stable under G.

Proof of Claim II: for any $h \in G$

$$\pi(h) \circ p_0 = \frac{1}{|G|} \sum_{g \in G} \pi(h)\pi(g) \circ p \circ \pi(g)^{-1}$$

$$= \frac{1}{|G|} \sum_{g \in G} \pi(hg) \circ p \cdot \pi(hg)^{-1} \cdot \pi(h)$$

$$\overset{g' = hg}{=} \frac{1}{|G|} \sum_{g' \in G} \pi(g') \cdot p \cdot \pi(g') \cdot \pi(h) = p_0 \cdot \pi(h).$$

Proof of Claim III: For any $v \in V$, $\pi(g)^{-1} v \in V$

$p \cdot \pi(g)^{-1} v \in W \Rightarrow \pi(g)\, p\, \pi(g)^{-1} v \in W$

$\Rightarrow p_0 v = \frac{1}{|G|} \sum_{g \in G} \pi(g)\, p\, \pi(g)^{-1} v \in W \Rightarrow \operatorname{Im}(p_0) \subseteq W.$

For any $w \in W$, $\pi(g^{-1}) w \in W \Rightarrow p\,\pi(g^{-1}) w = \pi(g^{-1}\, w$

$(\, p|_W = Id_W)$

$\Rightarrow \pi(g)\, p\, \pi(g)^{-1} w = w$

$\Rightarrow p_0 w = \frac{1}{|G|} \sum_{g \in G} \pi(g)\, p\, \pi(g)^{-1} w = w \Rightarrow \operatorname{Im}(p_0) \supseteq W.$

Note: this also shows: $p_0|_W = Id_W.$

Proof of Claim I: We first show $p_0 \cdot p = p.$

For $v \in V = W \oplus W'$, $\quad v = w + w'$

$$P_0 P(v) = P_0 w = w$$
$$P(v) = w \qquad \Rightarrow \quad P_0 P = P.$$

Then
$$P_0 \cdot P_0 = P_0 \cdot \frac{1}{|G|} \sum_{g \in G} \pi(g) P \pi(g)^{-1}$$
$$= \frac{1}{|G|} \sum_{g \in G} P_0 \cdot \pi(g) \cdot P \cdot \pi(g)^{-1}$$
$$= \frac{1}{|G|} \sum_{g \in G} \pi(g) P_0 \cdot P \pi(g)^{-1}$$
$$= \frac{1}{|G|} \sum_{g \in G} \pi(g) \cdot P \cdot \pi(g)^{-1} = P_0$$

This completes the proof of the claims and hence the theorem. $\square$