Definition: Let (π, V) be a repn of G. We say that it is irreducible (simple) if the only proper subrepn of V is $\{0\}$.

Remark: By the definition, the zero vector space is <u>not</u> irreducible.

Lemma: Let (π, V) be a repn of G and $\dim_{\mathbb{C}} V = 1$.

Then (π, V) is irreducible.

Proof: Let $W \subsetneq V$ be a subspace.

Since $\dim_{\mathbb{C}} V = 1$, $\dim_{\mathbb{C}} W = 0$.

Therefore, the only proper subrepn of (π, V) is $\{0\}$. $\square$

The following theorem is saying that the irreducible repns are "building blocks" for repns $/\mathbb{C}$.

Theorem. Every repn is a direct sum of irreducible repns.

Proof: Let (π, V) be a repn of G. Proof by induction.

If $\dim_{\mathbb{C}} V = 1$, this is irreducible and $V = V$.

Next, assume that $\dim_{\mathbb{C}} V = n+1$ and V is not irreducible.

Then there exists a subrepn $\{0\} \neq W \subsetneq V$.

By Maschke's theorem, we can find another $W_0 \subseteq V$

such that $V = W \oplus W_0$

and W, W_0 are subreprs of (π, V).

Since $\{0\} \neq W \subsetneq V$, $\dim W, \dim W_0 \leq n$.

Then by induction hypothesis, both W, W_0 can be

written as direct sums of irreducible repns,

$$W = U_1 \oplus U_2 \oplus \cdots U_r$$

$$W_0 = U_1' \oplus U_2' \oplus \cdots U_s'$$

Then $V = W \oplus W_0 = U_1 \oplus \cdots U_r \oplus U_1' \oplus U_2' \oplus \cdots U_s'$.

Then by induction, we complete the proof. $\square$

Observation. (A creterion for an irreducible repn)

Let (π, V) be a repn for G. Then (π, V) is irreducible if and only if, for any $0 \neq v \in V$,

$$V = \mathrm{span}_{\mathbb{C}} \{ \pi(g) v : g \in G \}$$

Proof: Let $v \in V$ nonzero. Set

$$W(v) = \mathrm{span}_{\mathbb{C}} \{ \pi(g) v : g \in G \}$$

This is a subrepn of V, since it is stable under G.

$(\Rightarrow)$ If (π, V) is irreducible, then $W(v) = V$ or $W(v) = \{0\}$

Sime $v \neq 0$, $\pi(g) v \neq 0$ ($\pi(g) : V \to V$ is an isomorphism)

Therefoe, $W(v) \neq \{0\} \Rightarrow W(v) = V$.

$(\Leftarrow)$ If (π, V) is not irreducible, then

$V = V_1 \oplus V_2 \oplus \cdots V_r$ with each V_i irreducible.

Take $v \in V_1$, then $W(v) \subseteq V_1$

This contradicts that $W(v) = V$. $\blacksquare$

Example: let $G = S_n$ and $V = \mathbb{C}^n$.

We defined the standard repn $(\mathrm{std}, \mathbb{C}^n)$ for S_n.

We want to write it as a direct sum of irreducible repns:

Recall $\mathbb{C}^n = \mathrm{span}_{\mathbb{C}} \{e_1, \dots e_n\}$ $\quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to i\text{-th}.$

We have two subspaces of $\mathbb{C}^n$:

$$W_1 = \{ \lambda e_1 + \lambda e_2 + \dots \lambda e_n : \lambda \in \mathbb{C} \} = \mathrm{span}_{\mathbb{C}} \{e_1 + \dots e_n\}.$$

$$W_2 = \{ \lambda_1 e_1 + \lambda_2 e_2 + \dots \lambda_n e_n : \lambda_1 + \dots \lambda_n = 0 \}.$$

Claim: the standard repn can be written as the direct sums of irreducible repns:

$$\mathbb{C}^n = W_1 \oplus W_2.$$

We need to show the following things:

(1) W_1 is stable under S_n.

Then $\dim_{\mathbb{C}} W_1 = 1 \Rightarrow W_1$ is irreducible

(2) W_2 is a vector space and

$$W_2 = \mathrm{span}_{\mathbb{C}} \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n\}.$$

(3) W_2 is stable under G and W_2 is irreducible.

We only prove (3): take $w \in W_2$, $w = \lambda_1 e_1 + \cdots \lambda_n e_n$

with $\lambda_1 + \cdots \lambda_n = 0$. Take $\sigma \in S_n$

$$\pi(\sigma) w = \lambda_1 e_{\sigma(1)} + \lambda_2 e_{\sigma(2)} + \cdots \lambda_n e_{\sigma(n)}$$

$$= \lambda_{\sigma^{-1}(1)} e_1 + \lambda_{\sigma^{-1}(2)} e_2 + \cdots \lambda_{\sigma^{-1}(n)} e_n$$

$\lambda_{\sigma^{-1}(1)} + \cdots \lambda_{\sigma^{-1}(n)} = \lambda_1 + \cdots \lambda_n = 0. \Rightarrow \pi(\sigma) w \in W_2.$

Next, we show that W_2 is irreducible. We use the creterion to show the irreducibility:

take $0 \neq w \in W_2$, $w = \lambda_1 e_1 + \lambda_2 e_2 + \cdots \lambda_n e_n$.

Since $\lambda_1 + \cdots \lambda_n = 0$ and $w \neq 0$, we can find $\underset{i \neq j}{\lambda_i \neq \lambda_j}$.

Take $\sigma = (i\ j)$

$$w = \lambda_1 e_1 + \cdots \lambda_i e_i + \cdots \lambda_j e_j + \cdots \lambda_n e_n$$

$$\pi(\sigma) w = \lambda_1 e_1 + \cdots \lambda_i e_j + \lambda_j e_i + \cdots \lambda_n e_n$$

$$\Rightarrow \quad w - \pi(\sigma) w = (\lambda_i - \lambda_j)(e_i - e_j) \in W_2.$$

$$\lambda_i \neq \lambda_j \Rightarrow e_i - e_j \in W_2.$$

$$\text{Take } \sigma' = \begin{pmatrix} i & j & \cdots \\ 1 & 2 & \cdots \end{pmatrix}$$

$$\pi(\sigma')(e_i - e_j) = e_1 - e_2 \in W_2.$$

Take $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad \pi(\sigma)(e_1 - e_2) = e_2 - e_3 \in W_2,$

We contimute this process and we can

$$e_1 - e_2, \ e_2 - e_3, \ e_3 - e_4 \cdots e_{n-1} - e_n \in W_2.$$

This shows $\quad \mathrm{span}_{\mathbb{C}} \{\pi(g) w_2 : g \in G\} = W_2.$

By the creterion, W_2 is irreducible. $\square$

Operations of representations.

$\boxed{\text{The dual repn}}$

Definition: Let V be a vector space. The <u>dual space</u> of V is:

$$V^* = \{ f : V \to \mathbb{C} : f \text{ are linear funtions} \}.$$

$$= \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

This is a vector space.

Moreover, let $\{v_1, \dots v_n\}$ be a basis for V.

Then we can define the linear funtions $v_1^*, \dots v_n^*$

as follows: for $v = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$

$$v_i^*(a_1 v_1 + a_2 v_2 + \cdots a_n v_n) = a_i.$$

Then $\{v_1^*, \cdots v_n^*\}$ will be a basis for V^*.

(This shows: $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V^*$ when $\dim_{\mathbb{C}} V < \infty$.)

Let (π, V) be a repn of G.

Definition: The <u>dual repn</u> (π^*, V^*) of G is defined as follows:

$$\pi^*: G \longrightarrow V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

$$g \longmapsto \pi^*(g)$$

$$(\pi^*(g) f)(v) = f(\pi(g^{-1}) v)$$

Check: $(\pi^*(g_1 g_2) f)(v) = f(\pi(g_2^{-1} g_1^{-1}) v)$

$$= f(\pi(g_2^{-1}) \pi(g_1^{-1}) v)$$

$$(\pi^*(g_1) \pi^*(g_2) f)(v) = \pi^*(g_1)(\pi^*(g_2) f)(v)$$

$$= (\pi^*(g_2) f)(\pi(g_1^{-1}) v) = f(\pi(g_2^{-1}) \pi(g_1^{-1}) v)$$

$\Rightarrow$ $\pi^*(g_1 g_2) f = \pi^*(g_1) \pi^*(g_2) f$ $\Rightarrow$ $\pi^*(g_1 g_2) = \pi^*(g_1) \pi^*(g_2)$

The tensor product

Definition: Let V, W be two vector spaces. The <u>tensor product</u>, denoted by $V \otimes W$, is the vector space of finite formal sums:

$$V \otimes W = \left\{ \sum \lambda_{i,j} \, v_i \otimes w_j \ : \ v_i \in V, \ w_j \in W \right\}$$

Satisfying:

(1) $\otimes$ is a bilinear map, that is: $a_1, a_2, b_1, b_2 \in \mathbb{C}$, $v_1, v_2 \in V$, $w_1, w_2 \in W$.

$$(a_1 v_1 + a_2 v_2) \otimes w = a_1 (v_1 \otimes w) + a_2 (v_2 \otimes w)$$

$$\begin{aligned} v \otimes (b_1 w_1 + b_2 w_2) &= v \otimes (b_1 w_1) + v \otimes (b_2 w_2) \\ &= b_1 (v \otimes w_1) + b_2 (v \otimes w_2). \end{aligned}$$

(2) suppose that V has a basis $\{v_1, \cdots v_n\}$
W has a basis $\{w_1, \cdots w_m\}$

then $V \otimes W$ has a basis $\{ v_i \otimes w_j \ : \ 1 \leq i \leq n, \ 1 \leq j \leq m \}$

(This shows: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$)

Definition: Let (π_1, V_1) and (π_2, V_2) be two repns of G.

Then the <u>tensor produt repn</u> is defined as follows:

$$\begin{aligned} \pi_1 \otimes \pi_2 : \quad & G \longrightarrow GL(V_1 \otimes V_2) \\ & g \longmapsto (\pi_1 \otimes \pi_2)(g) \end{aligned}$$

$$(\pi_1 \otimes \pi_2)(g)\Big(\sum \lambda_{i,j}\, v_i \otimes w_j\Big) = \sum \lambda_{i,j}\, (\pi_1(g)v_i) \otimes (\pi_2(g)v).$$

Example: Let (π_1, V_1) ; (π_2, V_2) be two repns of G.

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = \{\text{all linear maps} : V_1 \to V_2\}$$

This is a naturally vector space by defining:

$$(T_1 + T_2)(v) = T_1(v) + T_2(v)$$

$$(\lambda T)(v) = \lambda \cdot T(v) = T(\lambda v)$$

Then we can make it a G-repn: $(\tilde{\pi}, \mathrm{Hom}_{\mathbb{C}}(V_1, V_2))$

$$\begin{aligned} \tilde{\pi} : \quad & G \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \\ & g \mapsto \Big(T \xmapsto{\tilde{\pi}(g)} \pi_2(g) \circ T \circ \pi_1(g^{-1}) \Big) \end{aligned}$$

Exercise: $(\pi_1^* \otimes \pi_2, V_1^* \otimes V_2)$ and $(\tilde{\pi}, \mathrm{Hom}_G(V_1, V_2))$

as G- repns.

$\boxed{\text{Symmetric Square and Exterior Square.}}$

Let V be a vector space and we consider $V \otimes V$.

Let $\{v_1, \dots v_n\}$ be a basis for V.

Then we have an isomorphism $\theta : V \otimes V \longrightarrow V \otimes V$ by:

$$\theta(v_i \otimes v_j) = v_j \otimes v_i \quad \text{(and extend it linearly.)}$$

Notice that $\theta^2 = Id$.

Then we define:

$$\mathrm{Sym}^2(V) = \{ z \in V \otimes V : \theta(z) = z \}$$

$$= \mathrm{span}_{\mathbb{C}} \{ v_i \otimes v_j + v_j \otimes v_i : i \leq j \}$$

$$\Rightarrow \dim_{\mathbb{C}} \mathrm{Sym}^2(V) = \frac{n(n+1)}{2}$$

$$\Lambda^2(V) = \{ z \in V \otimes V : \theta(z) = -z \}$$

$$= \mathrm{span}_{\mathbb{C}} \{ v_i \otimes v_j - v_j \otimes v_i \quad i < j \}$$

$$\Rightarrow \dim_{\mathbb{C}} \Lambda^2(V) = \frac{n(n-1)}{2}$$

By comparing dimension: $\underset{\dim = n^2}{V \otimes V} = \underset{\dim = \frac{n(n+1)}{2}}{\mathrm{Sym}^2(V)} \oplus \underset{\dim \frac{n(n-1)}{2}}{\Lambda^2(V)}$

$\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$ are stable under G.

$\mathrm{Sym}^2(V)$: the symmetric square repn

$\Lambda^2(V)$: the exterior square repn.