Symmetric Square and Exterior Square.

Let V be a vector space and we consider $V \otimes V$.

Let $\{v_1, \dots v_n\}$ be a basis for V.

Then we have an isomorphism $\theta : V \otimes V \longrightarrow V \otimes V$ by:

$$\theta(v_i \otimes v_j) = v_j \otimes v_i \quad \text{(and extend it linearly.)}$$

Notice that $\theta^2 = Id$.

Then we define:

$$Sym^2(V) = \{ z \in V \otimes V : \theta(z) = z \}$$
$$= span_{\mathbb{C}} \{ v_i \otimes v_j + v_j \otimes v_i : i \leq j \}$$
$$\Rightarrow \dim_{\mathbb{C}} Sym^2(V) = \frac{n(n+1)}{2}$$

$$\Lambda^2(V) = \{ z \in V \otimes V : \theta(z) = -z \}$$
$$= span_{\mathbb{C}} \{ v_i \otimes v_j - v_j \otimes v_i \quad i < j \}$$
$$\Rightarrow \dim_{\mathbb{C}} \Lambda^2(V) = \frac{n(n-1)}{2}$$

By comparing dimension:
$$\underset{\dim = n^2}{V \otimes V} = \underset{\dim = \frac{n(n+1)}{2}}{Sym^2(V)} \oplus \underset{\dim \frac{n(n-1)}{2}}{\Lambda^2(V)}.$$

$Sym^2(V)$ and $\Lambda^2(V)$ are stable under G.

$Sym^2(V)$: the symmetric square repn

$\Lambda^2(V)$: the exterior square repn.

Let V be a vector space $/\mathbb{C}$. Let $T: V \to V$ be a linear map. We fix a basis $\{v_1, \ldots v_n\}$ of V.

Then

$$T(v_1, \ldots v_n) = (Tv_1, Tv_2, \ldots Tv_n) = (v_1, \ldots v_n)\, A.$$

with A an $n \times n$ matrix. $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

We define: $tr(A) = a_{11} + a_{22} + \cdots a_{nn} = \sum_{i=1}^{n} a_{ii}$.

As we vary basis, the matrix corresponding to T might change, but the trace will <u>never</u> change.

This is because: $tr(BAB^{-1}) = tr(A)$ for any invertible B.

Therefore: we *define* $tr(T) := tr(A)$.

Fact: Suppose that $T^m = Id$ for some $m \in \mathbb{Z}$. Then

T is a diagonalizable, that is, we can find a basis $\{v_1, \cdots v_n\}$ of V such that

$$T(v_1, \cdots v_n) = (Tv_1, \cdots Tv_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

λ_i are *eigenvalues* of T. $(Tv_i = \lambda_i v_i)$

In this case, we know $tr(T) = \lambda_1 + \cdots \lambda_n$.

Let (π, V) be a representation of G.

Definition: The *character* of (π, V) is a function

$\chi: G \to \mathbb{C}$ defined by:

$$\chi(g) := tr(\pi(g))$$

Lemma: Let (π, V) be a repn of G. Then for any $g \in G$,

$\pi(g)$ is diagonalizable.

Proof: This is because $\pi(g)^{|G|} = \pi(g^{|G|}) = \pi(e_g) = Id_V$.

Lemma: Let (π, V) be a repn of G. Then for any $g \in G$, the eigenvalues of $\pi(g)$ has modulous 1.

Proof: We know $\pi(g): V \to V$ is diagonalisable, then we can find a basis $\{v_1, \cdots v_n\}$ such that:

$$\pi(g)(v_1, \cdots v_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Set $m = |G|$, then $\pi(g)^m = Id$

$$\pi(g)^m (v_1, \cdots v_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{pmatrix}$$

$$Id\,(v_1, \cdots v_n) = (v_1 \cdots v_n)\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

This implies, $\lambda_i^m = 1. \Rightarrow |\lambda_i| = 1.$ □

Remark: We also show: each eigenvalue is a m-th root of unity, i.e. it is a root of the equation $z^m = 1$.

<u>Proposition:</u> Let (π, V) be a repn of G and χ its character. Suppose that $\dim_{\mathbb{C}} V = n$. Then:

(1) $\chi(e) = n$

(2) $\chi(g^{-1}) = \overline{\chi(g)}$

(3) $\chi(g_1 g_2 g_1^{-1}) = \chi(g_2)$ (or $\chi(g_1 g_2) = \chi(g_2, g_1)$)

Proof: (1) $\chi(e) = \operatorname{tr}(\pi(e)) = \operatorname{tr}(Id_V) = n.$

(2) $\chi(g^{-1}) = \operatorname{tr}(\pi(g^{-1})) = \operatorname{tr}(\pi(g)^{-1})$

If $\pi(g)(v_1, \cdots v_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

then $\pi(g)^{-1}(v_1, \cdots v_n) = (v_1, \cdots v_n) = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}$

$\Rightarrow \chi(g^{-1}) = \lambda_1^{-1} + \cdots \lambda_n^{-1}$

λ_i has modulous 1 $\Rightarrow |\lambda_i| = 1 \Rightarrow \lambda_i^{-1} = \overline{\lambda_i}$

$\Rightarrow \chi(g^{-1}) = \overline{\lambda_1} + \overline{\lambda_2} + \cdots \overline{\lambda_n} = \overline{\chi(g)}.$

(3) $\chi(g_1 g_2 g_1^{-1}) = \operatorname{tr}(\pi(g_1 g_2 g_1^{-1}))$

$= \operatorname{tr}(\pi(g_1)\pi(g_2)\pi(g_1)^{-1}) = \operatorname{tr}(\pi(g_2))$

$= \chi(g_2).$ □

Example: Let (π_1, V_1) and (π_2, V_2) be two repns of G with charaters χ_1, χ_2. Then:

(1) The direct sum $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ has characters

$$\chi_{V_1 \oplus V_2} = \chi_1 + \chi_2 .$$

(2) The tensor product $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ has charaters

$$\chi_{V_1 \otimes V_2} = \chi_1 \cdot \chi_2$$

Proof: Take $g \in G$. Let $\{v_1, \cdots v_n\}$ be a basis for V_1

$\{v_1', \cdots v_m'\}$ be a basis for V_2

$$\pi_1(g)(v_1 \cdots v_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad \chi_1(g) = \lambda_1 + \cdots \lambda_n$$

$$\pi_2(g)(v_1' \cdots v_m') = (v_1', \cdots, v_m') = \begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_m' \end{pmatrix} \qquad \chi_2(g) = \lambda_1' + \cdots \lambda_m'$$

(1) We know $\{v_1, \cdots v_n, v_1', \cdots v_m'\}$ is a basis for $V_1 \oplus V_2$

and $\pi_1 \oplus \pi_2(g)\left(v_1, \cdots v_n, v_1' \cdots v_m'\right)$

$$= \left(\pi_1(g) v_1, \cdots \pi_1(g) v_n, \pi_2(g) v_1' \cdots \pi_2(g) v_m'\right)$$

$$= (v_1 \cdots v_n, v_1', \cdots v_m') = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \lambda_1' & & \\ & & & & \ddots & \\ & & & & & \lambda_m' \end{pmatrix}$$

$$\Rightarrow \chi_{V_1 \oplus V_2}(g) = tr(\pi_1 \oplus \pi_2 (g)) = \lambda_1 + \cdots \lambda_n + \lambda_1' + \cdots \lambda_m'$$
$$= \chi_1(g) + \chi_2(g)$$

(2) By the construction of $V_1 \otimes V_2$,

$\{ v_i \otimes v_j' : 1 \leq i \leq n, 1 \leq j \leq m \}$ is a basis for $V_1 \otimes V_2$

$$\pi_1 \otimes \pi_2 (g) (v_i \otimes v_j') = (\pi_1(g) v_i) \otimes (\pi_2(g) v_j')$$
$$= (\lambda_i v_i) \otimes (\lambda_j' v_j') = \lambda_i \lambda_j' v_i \otimes v_j'$$

$$\Rightarrow \chi_{V_1 \otimes V_2}(g) = tr(\pi_1 \otimes \pi_2 (g))$$
$$= \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \lambda_i \lambda_j'$$
$$= (\lambda_1 + \cdots \lambda_n)(\lambda_1' + \cdots \lambda_m') = \chi_1(g) \chi_2(g). \quad \square$$

Example: Let (π, V) be a repn of G and χ its character.

In the previous lecture, we defined the $\mathrm{Sym}^2 V$ repn and $\Lambda^2 V$ repn, which are subrepns of $V \otimes V$.

Denote by χ_{sym^2}, χ_{Λ^2} their characters.

Then:

$$\chi_{\mathrm{sym}^2}(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$$

and $$\chi_{\Lambda^2}(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$$

Proof: Only prove $\chi_{\mathrm{sym}^2}(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$.

Again, we choose the basis $\{v_1, \cdots v_n\}$ of V such that:

$$\pi(g)(v_1 \cdots v_n) = (v_1, \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

This implies: $\pi(g^2)(v_1, \cdots v_n) = (v_1, \cdots v_n) = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix}$

By the basis of V, we have a basis of $\mathrm{Sym}^2 V$:

$$v_i \otimes v_j + v_j \otimes v_i \quad \text{with } 1 \le i \le j \le n.$$

$$\begin{aligned} \pi\otimes\pi(g)&\left(v_i \otimes v_j + v_j \otimes v_i\right) \\ &= \left(\pi(g) v_i\right) \otimes \left(\pi(g) v_j\right) + \left(\pi(g) v_j\right) \otimes \left(\pi(g) v_i\right) \\ &= \lambda_i \lambda_j\, v_i \otimes v_j + \lambda_j \lambda_i\, v_j \otimes v_i \\ &= \lambda_i \lambda_j \left(v_i \otimes v_j + v_j \otimes v_i\right) \end{aligned}$$

$$\Rightarrow \chi_{\mathrm{sym}^2}(g) = \sum_{1 \le i \le j \le n} \lambda_i \lambda_j$$

On the other hand,

$$\frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) = \frac{1}{2}\left((\lambda_1 + \cdots \lambda_n)^2 + \lambda_1^2 + \lambda_2^2 + \cdots \lambda_n^2 \right)$$

$$= \frac{1}{2} \cdot 2 \sum_{1 \le i \le j \le n} \lambda_i \lambda_j = \sum_{1 \le i \le j \le n} \lambda_i \lambda_j \qquad \square$$

A discussion on the m-th root of unity:

Let $m \geq 1$ be an integer. We consider the solution of the equation:

$$X^m = 1.$$

Suppose that $z = re^{i\theta} \in \mathbb{C}$ and $z^m = 1$,

$z^m = 1 \Rightarrow |z| = 1 \Rightarrow r = 1.$

Therefore, $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

$$z^m = (e^{i\theta})^m = e^{im\theta} = 1 = e^{i \cdot 2\pi k} \quad k \in \mathbb{Z}.$$

This means: $m\theta = 2\pi \cdot k$

and hence $\theta = \frac{2\pi k}{m}$

$k = 0, \quad \theta = 0$

$k = 1 \quad \theta = \frac{2\pi}{m}$

$k = 2, \quad \theta = \frac{2\pi \cdot 2}{m}$

$\vdots$

$k = m-1$, $\theta = \dfrac{2\pi(m-1)}{m}$

$k = m$ $\quad \theta = \dfrac{2\pi \cdot m}{m} = 2\pi = 0 + 2\pi$

We know: $e^{i(\theta+2\pi)} = e^{i\theta}$

Therefore, we have m solutions:

$$1,\ e^{i\frac{2\pi}{m}},\ e^{i\cdot\frac{4\pi}{m}},\ \cdots\ e^{i\frac{2(m-1)\pi}{m}}$$

or $\left\{ e^{i\cdot\frac{2\pi k}{m}} : 0 \le k \le m-1 \right\}$