Shur's Lemma: Let (π_1, V_1) and (π_2, V_2) be two irreducible repns of G. Let $T: V_1 \to V_2$ be an intertwining operator.

Then:

(1) If (π_1, V_1) and (π_2, V_2) are not isomorphic, then $T=0$.

(2) If $(\pi_1, V_1) = (\pi_2, V_2)$, then we can find $\lambda \in \mathbb{C}$ such that $T = \lambda \cdot Id_V$.

Proof: (1): Suppose that $0 \neq T: V_1 \to V_2$ is an intertwining operator.

Then $\ker T$ is a subrepn of V_1.

V_1 is irreducible $\Rightarrow$ $\ker(T) = \{0\}$ or $\ker(T) = V_1$.

$T \neq 0 \Rightarrow \ker(T) = \{0\} \Rightarrow T$ is injective.

On the other hand, $\operatorname{Im}(T)$ is a subrepn of V_2.

V_2 irreducible $\Rightarrow \operatorname{Im}(T) = \{0\}$ or $\operatorname{Im}(T) = V_2$

$T \neq 0 \Rightarrow \operatorname{Im}(T) = V_2 \Rightarrow T$ is surjective.

This means: T is a bijective intertwining operator and hence V_1 is isomorphic to V_2, a contradiction.

(2) Suppose that $T: V_1 \to V_1$

Let λ be an eigenvalue for T.

Set $V_\lambda = \{v \in V : Tv = \lambda v\} \neq \{0\}$ and $T_\lambda = T - \lambda \mathrm{Id}_V$.

T is an intertwining operator $\Rightarrow$ T_λ is an intertwinig operator.

We know: V_1 is irreducible $\Rightarrow$ $\ker(T_\lambda) = \{0\}$ or $\ker\{T_\lambda\} = V_1$

We can show: $\ker(T_\lambda) = V_\lambda \neq \{0\}$ $\Rightarrow$

$$\ker(T_\lambda) = V_\lambda = V_1$$

This concludes: $Tv = \lambda v$. for any $v \in V_1$ □

In the following, we assume (π_1, V_1) and (π_2, V_2) are irreducible repns of G and $|G|$ is the order of G.

Corollary: Let $h : V_1 \to V_2$ be a linear map. Set:

$$h_0 = \frac{1}{|G|} \sum_{g \in G} \pi_2(g^{-1}) \cdot h \cdot \pi_1(g)$$

Then: (1) If (π_1, V_1) and (π_2, V_2) are not isomorphic, then $h_0 = 0$.

(2) If $(\pi_1, V_1) = (\pi_2, V_2)$, then

$$h_0 = \lambda \cdot \mathrm{Id}_{V_1} \quad \text{with} \quad \lambda = \frac{tr(h)}{n}$$

$n = \dim_{\mathbb{C}} V_1$.

Proof: Observation: h_0 is an intertwining operator.

(1) If $(\pi_1, V_1) \ncong (\pi_2, V_2)$, $h_0 = 0$ by Schur's Lemma.

(2) If $(\pi_1, V_1) = (\pi_2, V_2)$, $h_0 = \lambda \cdot Id_{V_1}$ by Schur's Lemma.

(If we have a matrix $A = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$, then $\lambda = \frac{tr(A)}{n}$)

$$tr(h_0) = tr\left(\frac{1}{|G|} \sum_{g \in G} \pi_1(g^{-1}) \cdot h \cdot \pi_1(g) \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} tr\left(\pi_1(g^{-1}) \cdot h \cdot \pi_1(g) \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} tr(h) = tr(h)$$

$$\Rightarrow \quad h_0 = \lambda \, Id_V \quad \text{with} \begin{cases} \lambda = \frac{tr(h)}{n} \\ n = \dim_{\mathbb{C}} V \end{cases}$$ □

Next, we fix a basis $\{v_1, \dots v_n\}$ for V_1

a basis $\{v_1', \dots v_m'\}$ for V_2.

Then for $g \in G$, $\pi_1(g) = \left(M_{ij}(g) \right)_{1 \le i, j \le n}$

$\pi_2(g) = \left(N_{kl}(g) \right)_{1 \le k, l \le m}$

Notice: $\chi_1(g) = M_{11}(g) + M_{22}(g) + \cdots M_{nn}(g)$

$\chi_2(g) = N_{11}(g) + N_{22}(g) \cdots N_{mm}(g)$

Remark: $M_{ij}(g)$ is called the matrix coefficient of a representation.

Set $h = (x_{\ell i})_{\substack{1 \le \ell \le m \\ 1 \le i \le n}}$. Then: $h_0 = (y_{kj})_{\substack{1 \le k \le m \\ 1 \le j \le n}}$

$$y_{kj} = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{1 \le \ell \le m \\ 1 \le i \le n}} N_{k\ell}(g^{-1}) \cdot x_{\ell i} \cdot M_{ij}(g)$$

Case I: If $(\pi_1, V_1) \ncong (\pi_2, V_2)$, then $h_0 = 0$

and hence $y_{kj} = 0$ for all k, j.

Since we are choosing $x_{\ell i}$ arbitrarily, we have:

$$(*) \quad \frac{1}{|G|} \sum_{g \in G} N_{k\ell}(g^{-1}) M_{ij}(g) = 0.$$

Case II: Assume that $(\pi_1, V_1) = (\pi_2, V_2)$, then

$$h_0 = \lambda \mathrm{Id}_{V_1} = \frac{\mathrm{tr}(h)}{n} \cdot \mathrm{Id}_{V_1}$$ Set $\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$

$$\lambda = \frac{1}{n} \sum_{1 \le \ell, i \le n} \delta_{\ell i} x_{\ell i}$$

$$\Rightarrow \quad y_{kj} = \frac{1}{n} \sum_{1 \le l, i \le n} \delta_{li} X_{li} \cdot \delta_{kj}.$$

$$\frac{1}{|G|} \sum_{g \in G} \sum_{1 \le l, i \le n} N_{kl}(g^{-1}) X_{li} M_{ij}(g)$$

$$= \frac{1}{n} \sum_{1 \le l, i \le n} \delta_{li} \delta_{kj} X_{li}.$$

that is:

$$\sum_{1 \le l, i \le n} \left(\frac{1}{|G|} \sum_{g \in G} N_{kl}(g^{-1}) M_{ij}(g) \right)$$

$$= \sum_{1 \le l, i \le n} \left(\frac{\delta_{li} \delta_{kj}}{n} \right) X_{li}.$$

By equating X_{li},

$$\Rightarrow \frac{1}{|G|} \sum_{g \in G} N_{kl}(g^{-1}) M_{ij}(g) = \frac{\delta_{li} \delta_{kj}}{n}$$

Corollary: Assume the notations above

(1) If $V_1 \ncong V_2$, then

$$\frac{1}{|G|} \sum_{g \in G} N_{kl}(g^{-1}) M_{ij}(g) = 0$$

(2) If $V_1 = V_2$ and $\pi_1 = \pi_2$, then

$$\frac{1}{|G|}\sum_{g\in G} N_{kl}(g^{-1}) M_{ij}(g) = \frac{\delta_{li}\delta_{kj}}{n}$$

$$\boxed{\text{(1st) Orthogonality relations}}$$

Let ϕ, ψ be two complex-valued funtions on G, that is:

$$\phi : G \to \mathbb{C}, \quad \psi : G \to \mathbb{C}.$$

We can define the scalar product:

$$(\phi|\psi) = \frac{1}{|G|}\sum_{g\in G} \phi(g)\overline{\psi(g)}$$

Check: (1) $(\lambda_1\phi_1 + \lambda_2\phi_2 | \psi) = \lambda_1(\phi_1|\psi) + \lambda_2(\phi_2|\psi)$

(2) $(\phi| \lambda_1\psi_1 + \lambda_2\psi_2) = \overline{\lambda_1}(\phi|\psi_1) + \overline{\lambda_2}(\phi|\psi_2)$

(3) $(\phi|\psi) = \overline{(\psi, \phi)}$

(4) $(\phi|\phi) \geq 0$ for any ϕ

$(\phi|\phi) = 0$ if and only if $\phi = 0$.

Proposition: (1) Let χ be the character of an irreducible repn, then $(\chi|\chi) = 1$

(2) Let χ_1, χ_2 be the characters of two non isomorphic repns (π_1, V_1) and (π_2, V_2) then $(\chi_1 | \chi_2) = 0$.

Proof: We fix a basis $\{v_1, \cdots v_n\}$ for V_1

$\{v_1', \cdots v_m'\}$ for V_2.

Then
$$\pi_1(g) = (M_{ij}(g))_{1 \le i, j \le n}$$
$$\pi_2(g) = (N_{kl}(g))_{1 \le k, l \le m}$$

(1) Take $\chi = \chi_1$, the character of (π_1, V_1).

$$(\chi_1|\chi_1) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_1(g)}$$
$$= \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_1(g^{-1})$$
$$= \frac{1}{|G|} \sum_{g \in G} (M_{11}(g^{-1}) + \cdots M_{nn}(g^{-1}))(M_{11}(g) + \cdots M_{nn}(g))$$
$$= \sum_{1 \le i, j \le n} \frac{1}{|G|} \sum_{g \in G} M_{ii}(g^{-1}) M_{jj}(g)$$

$$= \sum_{1 \le i, j \le n} \frac{1}{n} \delta_{ij} \delta_{ji} = 1.$$

(2) $$(\chi_1 | \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1})$$

$$= \sum_{\substack{1 \le i \le n \\ 1 \le l \le m}} \frac{1}{|G|} \sum_{g \in G} N_{ll}(g^{-1}) M_{ii}(g) = 0$$

Let (π, V) be a repn of G. Then

$(\pi, V) = (\pi_1, V_1) \oplus \cdots (\pi_k, V_k)$ with V_i irreducible.

Furthermore, we can rearrange V_i such that

$(\pi, V) = m_1 (\pi_1, V_1) \oplus m_2 (\pi_2, V_2) \oplus \cdots m_r (\pi_r, V_r)$.

such that distinct V_i, V_j are non-isomorphic

Furthermore, if we allow $m_i = 0$,

$$(\pi, V) = \bigoplus_{W \text{ irred}} m_\rho (\rho, W)$$

Here the direct sum is over all irreducible repns of G.

We use the notation $\mathrm{Irr}(G)$ for all (non-isomorphic) repns of G.

Definition: m_ρ is called the index of (ρ, W) in (π, V).

We use the notation $\langle \pi, \rho \rangle := m_\rho$.

Theorem: Let (π, V) be a repn of G with character ϕ

Let (ρ, W) be an irreducible repn of G with character χ. Then

$$\langle \pi, \rho \rangle = \langle \phi | \chi \rangle$$

Proof: Suppose that $(\pi, V) = \bigoplus m_\rho (\rho, W)$

Then $$\phi = \sum_{W \in Irr(G)} m_\rho \cdot \chi_W$$

Then:
$$\langle \phi | \chi \rangle = \sum_{W \in Irr(G)} m_\rho \langle \chi_W | \chi \rangle = m_\rho. \quad \square.$$

Corollary: Two repns with the same character are isomorphic.

Proof: $$(\pi_1, V_1) = \bigoplus_{W \in Irr(G)} m_W (\rho, W) \rightsquigarrow \phi_1$$

$$(\pi_2, V_2) = \bigoplus_{W \in Irr(G)} n_W (\rho, W) \rightsquigarrow \phi_2$$

$$\phi_1 = \phi_2 \Rightarrow m_W = n_W \text{ for all } W \in Irr(G). \quad \square$$

Theorem: Let ϕ be the character for a repn (π, V)

(1) $(\phi | \phi)$ is always an integer.

(2) $(\phi | \phi) = 1$ if and only if (π, V) is irreducible.

Proof: (1) $(\pi, V) = \bigoplus_{W \in \mathrm{Irr}(G)} m_W (\rho, W) \rightsquigarrow \phi$

$$(\phi | \phi) = \left(\sum_W m_W \chi_W \,\Big|\, \sum_{W'} m_{W'} \chi_{W'} \right)$$

$$= \sum_{W, W' \in \mathrm{Irr}(G)} m_W m_{W'} (\chi_W | \chi_{W'})$$

$$= \sum_{W \in \mathrm{Irr}(G)} m_W^2 .$$

(2) ($\Rightarrow$) $(\phi | \phi) = 1 \Rightarrow$ only 1 $m_W = 1$ and others $= 0$.

$\Rightarrow (\pi, V) = (\rho, W)$ for some $W \in \mathrm{Irr}(G)$.

($\Leftarrow$) If (π, V) is irreducible, then $(\phi | \phi) = 1$. $\blacksquare$