Recall:

Proposition: (1) Let χ be the character of an irreducible repn, then $(\chi|\chi) = 1$

(2) Let χ_1, χ_2 be the characters of two non isomorphic repns (π_1, V_1) and (π_2, V_2) then $(\chi_1|\chi_2) = 0$.

Let (π, V) be a repn of G. Then

$$(\pi, V) = (\pi_1, V_1) \oplus \cdots (\pi_k, V_k) \quad \text{with } V_i \text{ irreducible.}$$

Furthermore, we can rearrage V_i such that

$$(\pi, V) = m_1(\pi_1, V_1) \oplus m_2(\pi_2, V_2) \oplus \cdots m_r(\pi_r, V_r)$$

such that distinct V_i, V_j are non-isomorphic

Furthemore, if we allow $m_i = 0$,

$$(\pi, V) = \bigoplus_{W \text{ irred}} m_\rho (\rho, W)$$

Here the direct sum is over all irreducible repns of G.

We use the notation $\mathrm{Irr}(G)$ for all (non-isomorphic) repns of G.

Definition: m_ρ is called the index/multiplicity of (ρ, W) in (π, V).

We use the notation $\langle \pi, \rho \rangle := m_\rho$.

Theorem: Let (π, V) be a repn of G with character ϕ

Let (ρ, W) be an irreducible repn of G with character χ Then

$$\langle \pi, \rho \rangle = \langle \phi | \chi \rangle$$

Proof: Suppose that $(\pi, V) = \bigoplus m_\rho (\rho, W)$

Then $\phi = \sum_{W \in Irr(G)} m_\rho \cdot \chi_W$

Then:

$$\langle \phi | \chi \rangle = \sum_{W \in Irr(G)} m_\rho \langle \chi_W | \chi \rangle = m_\rho. \quad \square.$$

Corollary: Two repns with the same characte are isomorphic.

Proof: $(\pi_1, V_1) = \bigoplus_{W \in Irr(G)} m_W (\rho, W) \rightsquigarrow \phi_1$

$(\pi_2, V_2) = \bigoplus_{W \in Irr(G)} n_W (\rho, W) \rightsquigarrow \phi_2$

$\phi_1 = \phi_2 \Rightarrow m_W = n_W$ for all $W \in Irr(G)$. $\square$

Theorem: Let ϕ be the character for a repn (π, V)

(1) $(\phi|\phi)$ is always an integer.

(2) $(\phi|\phi) = 1$ if and only if (π, V) is irreducible.

Proof: (1) $(\pi, V) = \bigoplus_{W \in Irr(G)} m_W (\rho, W) \rightsquigarrow \phi$

$$(\phi|\phi) = \left(\sum_W m_W \chi_W \,\middle|\, \sum_{W'} m_{W'} \chi_{W'} \right)$$

$$= \sum_{W, W' \in Irr(G)} m_W m_{W'} (\chi_W | \chi_{W'})$$

$$= \sum_{W \in Irr(G)} m_W^2 .$$

(2) $(\Rightarrow)$ $(\phi|\phi) = 1 \Rightarrow$ only 1 $m_W = 1$ and others $= 0$.

$\Rightarrow (\pi, V) = (\rho, W)$ for some $W \in Irr(G)$.

$(\Leftarrow)$ If (π, V) is irreducible, then $(\phi|\phi) = 1$. $\blacksquare$

Recall: the regular representation:

$G = G \qquad V = \mathbb{C}[G] = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\}$

$$R(h)\left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g (hg)$$

We know: $\left(R, \mathbb{C}[G]\right) = \bigoplus_{W \in \mathrm{Irr}(G)} m_\rho \cdot (\rho, W)$

Theorem: For each $W \in \mathrm{Irr}(G)$, $m_\rho = \dim_{\mathbb{C}} W \ (\neq 0)$

Proof: Let r_G be the character of $(R, \mathbb{C}[G])$.

Claim: $r_G(e) = |G|$

$r_G(g) = 0$ if $g \neq e$.

Proof of Claim: $r_G(e) = tr(\pi(e)) = tr(Id_{\mathbb{C}[G]})$

$= \dim_{\mathbb{C}} \mathbb{C}[G] = |G|$

$g \neq e \quad r_G(g) = tr(\pi(g))$

Notice that: for any $h \in G$, $e \neq g \in G$, $gh \neq h$.

$\Rightarrow \quad tr(\pi(g)) = 0 \Rightarrow r_G(g) = 0.$

We know $\quad m_\rho = \langle r_G | \chi_\rho \rangle \quad$ (χ_ρ is the character of (ρ, W))

$$\langle r_G | \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in G} r_G(g) \chi_\rho(g)$$

$$= \frac{1}{|G|} r_G(e) \chi_\rho(e) = \frac{1}{|G|} \cdot |G| \cdot \dim_{\mathbb{C}} W$$

$$= \dim_{\mathbb{C}} W.$$

Next, denote $\mathrm{Irr}(G)$ the set of all non-isomorphic irreducible repns. For each $(\rho, W) \in \mathrm{Irr}(G)$, set $n_W = \dim_{\mathbb{C}} W$. and χ_W its character

Corollary: (a) $|G| = \sum_{W \in \mathrm{Irr}(G)} n_W^2$

(b) If $e \neq g \in G$, then $\sum_{W \in \mathrm{Irr}(G)} n_W \chi_W(g) = 0$.

Proof: We know : $(R, \mathbb{C}[G]) = \bigoplus_{W \in \mathrm{Irr}(G)} n_W (\rho, W)$

Let r_G be the character, then

$$r_G = \sum_{W \in \mathrm{Irr}(G)} n_W \chi_W$$

(1) Take $g = e$, $r_G(e) = |G|$

$\chi_W(e) = \mathrm{tr}(\rho(e)) = \dim_{\mathbb{C}} W = n_W$

$$\Rightarrow |G| = \sum_{W \in \mathrm{Irr}(G)} n_W^2$$

(2) Take $g \neq e$, $r_G(g) = 0$

$$\Rightarrow \sum_{W \in \mathrm{Irr}(G)} n_W \chi_W(g) = 0.$$

Remark: This corollary also implies:

there are <u>finitely</u> many non-isomorphic irreducible repns.

$\boxed{\text{Question:}}$: What is the number of non-isomorphic irreducible repns?

Definition: Let G be a group. A funtion $f: G \to \mathbb{C}$ is a <u>class function</u> if

$$f(ghg^{-1}) = f(h) \quad \text{for all } g, h \in G$$

Example: Let χ be a charater for a repn (π, V) of G.

Then χ is a class funtion.

We define:

$$H := \{ f: G \to \mathbb{C},\ f \text{ is a class funtion} \}.$$

It is easy to show: this is a vector space $/\mathbb{C}$.

Indeed, this is a finite-dimensional vector space, and we want to compute its basis.

Definition: Let g_1, g_2 be two elements in G. We say they are <u>conjugate</u> if we can find $g \in G$ such that

$$g g_1 g_1^{-1} = g_2.$$

This defines an equivalent relation on G, which partitions G into (conjugate) classes.: that is

$$G = \bigsqcup_{[g] \in C(G)} \{ hgh^{-1} : h \in G \}$$

Here $C(G)$ is the set of all nonequivalent classes.

We can construct a basis for G by $C(G)$.:

for $[g] \in C(G)$, we define

$$f_{[g]}(h) = \begin{cases} 1 & \text{if } h \text{ and } g \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise: $\{ f_{[g]} : [g] \in C(G) \}$ will form a basis for H and hence $\dim_{\mathbb{C}} H = \# C(G)$, the number of non equivalent conjugate classes.

On the other hand, we show:

$\{ \overline{\chi_W} : W \in Irr(G) \}$ form an orthonormal basis for H.

This implies: the number of non-isomorphic irreducible repns

$= \dim_{\mathbb{C}} H$

$=$ the number of non equivalent conjugate classes of G.

Theorem: $\{ \overline{\chi_W} : W \in Irr(G) \}$ forms an orthonormal basis for H.

Proposition: Let $f: G \to \mathbb{C}$ be a class funtion. and (π, V) be a repn of G. Then we can define a linear map:

$$\pi(f) = \sum_{g \in G} f(g)\, \pi(g) : V \to V.$$

If V is irreducible and $\dim_{\mathbb{C}} V = n$, then $\pi(f) = \lambda \, Id_V$ and $\lambda = \frac{|G|}{n} \langle f | \overline{\chi} \rangle$ Here χ is the characten of V.

(Note: if $(\pi, V) = (\pi_1, V_1) \oplus (\pi_2, V_2)$, then $\pi(f) = \pi_1(f) + \pi_2(f)$)

Proof: we first show that $\pi(f)$ is an intertwining operator

For $h \in G$, $\pi(h)\pi(f)\pi(h^{-1}) = \pi(h)\left(\sum_{g\in G} f(g)\pi(g)\ \pi(h^{-1})\right)$

$$= \sum_{g\in G} f(g)\ \pi(hgh^{-1})$$

$$= \sum_{g\in G} f(hgh^{-1})\pi(hgh^{-1})$$

$$= \sum_{g\in G} f(g)\pi(g) = \pi(f).$$

Sime V is irreduible, then $\pi(f) = \lambda\, Id_V$ (by Schur's Lemma)

Then: $\lambda = \frac{1}{n} tr(\pi(f)) = \frac{1}{n}\ tr\left(\sum_{g\in G} f(g)\pi(g)\right)$

$$= \frac{1}{n}\cdot \sum_{g\in G} f(g)\ tr(\pi(g))$$

$$= \frac{1}{n}\sum_{g\in G} f(g)\ \chi(g) = \frac{|G|}{n} \langle f | \overline{\chi} \rangle$$

A.

Proof of Theorem: By previous lectures, we know

$\{\overline{\chi_W} : W \in Irr(G)\}$ are orthonormal.

So it sufficies to show they generate H.

Proof by contradiction. Suppose not. Then we can find

$f_0 \in H$ such that $\langle f_0 | \overline{\chi_W} \rangle = 0$ for all $W \in Irr(G)$

(This is because $\langle \cdot | \cdot \rangle$ is an inner form on H and hence we can consider the orthogonal decomposition.)

Let (ρ, W) be an irreducible repn of G, we consider

$$\rho(f) : W \to W$$

We know $\rho(f) = \frac{|G|}{\dim W} \cdot \langle f | \overline{\chi_W} \rangle = 0$

Therefore $\rho(f)$ is always the zero map.

Next, we consider $(R, \mathbb{C}[G])$

Since $(R, \mathbb{C}[G]) = \bigoplus_{W \in Irr(G)} \dim W \cdot (\rho, W)$

$$\Rightarrow R(f) = \sum_{W \in Irr(G)} \dim W \cdot \rho(f) = 0.$$

By definition: $0 = R(f)\, e = \left(\sum_{g \in G} f(g) R(g) \right)(e)$

$$= \sum_{g \in G} f(g) (ge) = \sum_{g \in G} f(g)\, g$$

This forces: $f(g) = 0$ for all $g \in G$. $\square$