Recall: in the last lecture, we showed there are finitely many (non-isomorphic) irreducible repns.

Denote by $\mathrm{Irr}(G)$ the number of (non-isomorphic) irred repns of the group G.

<u>Goal</u>: $\# \mathrm{Irr}(G)$ = the number of (inequivalent) conjugate classes of G.

To show this, we introduce the vector space of <u>class functions</u>:

$$H = \{ f : G \to \mathbb{C} \ : \ f(g_1 g_2 g_1^{-1}) = f(g_2) \text{ for all } g_1, g_2 \}$$

We can show: $\dim_{\mathbb{C}} H$ = the number of (inequivalent) conjugate classes of G.

On the other hand, for $(\rho, W) \in \mathrm{Irr}(G)$, denote by χ_W its character.

Then $\{ \overline{\chi_W} : W \in \mathrm{Irr}(G) \} \subseteq H$ and is an orthonormal set w.r.t $(\ )$.

Therefore, it suffices to show:

$$H = \mathrm{span}_{\mathbb{C}} \{ \overline{\chi_W} : W \in \mathrm{Irr}(G) \}$$

Last Time:

Proposition: Let $f: G \to \mathbb{C}$ be a class funtion. and (π, V) be a repn of G. Then we can define a linear map:

$$\pi(f) = \sum_{g \in G} f(g)\, \pi(g) : V \to V.$$

(1) If $(\pi, V) = (\pi_1, V_1) \oplus (\pi_2, V_2)$, then $\pi(f) = \pi_1(f) + \pi_2(f)$

(2) If V is irreducible and $\dim_{\mathbb{C}} V = n$, then $\pi(f) = \lambda \, Id_V$ and $\lambda = \frac{|G|}{n} \langle f | \overline{\chi} \rangle$. Here χ is the character of V.

Theorem: $\{\overline{\chi_W} : W \in Irr(G)\}$ spans H and hence $\{\overline{\chi_W} : W \in Irr(G)\}$ is an orthormal basis for H.

Proof: (By contradiction): Suppose not. Then we can find $f_0 \in H$ such that $\langle f_0 | \overline{\chi_W} \rangle = 0$ for all $W \in Irr(G)$

Let (ρ, W) be an irreducible repn of G, we consider

$$\rho(f) : W \to W$$

We know $\rho(f) = \frac{|G|}{\dim W} \cdot \langle f | \overline{\chi_W} \rangle \, Id_W = 0$

Therefore $\rho(f)$ is always the zero map.

Next, we consider $(R, \mathbb{C}[G])$

$$\text{Since } (R, \mathbb{C}[G]) = \bigoplus_{W \in Irr(G)} \dim W \cdot (\rho, W)$$

$$\Rightarrow R(f) = \sum_{W \in Irr(G)} \dim W \cdot \rho(f) = 0.$$

By definition: $0 = R(f)\, e = \left(\sum_{g \in G} f(g) R(g) \right)(e)$

$$= \sum_{g \in G} f(g) (ge) = \sum_{g \in G} f(g)\, g$$

This forces: $f(g) = 0$ for all $g \in G$. $\square$

This completes the proof:

$\# Irr(G) = \#$ of (inequivalent) conjugate classes of G.

We set both to be h. $\{\chi_W : W \in Irr(G)\} = \{\chi_1, \cdots \chi_h\}$

Corollary: Let $g \in G$, and $c(g)$ the number of elements in the conjugate class $[g]$

(1) $$\sum_{i=1}^{h} \chi_i(g) \overline{\chi_i(g)} = \frac{|G|}{c(g)}$$

(2) If g' is not conjugate to g, then:

$$\sum_{i=1}^{h} \chi_i(g)\,\overline{\chi_i(g')} = 0.$$

Proof: Let $f_{[g]}(g') = \begin{cases} 1 & g' \text{ is conjugate to } g \\ 0 & \text{otherwise} \end{cases} \in H$

$\{\overline{\chi_W} : W \in \mathrm{Irr}(G)\}$ is an orthonormal basis

$\Rightarrow \{\chi_W : W \in \mathrm{Irr}(G)\}$ is an orthonormal basis

$$\Rightarrow f_{[g]} = \sum_{i=1}^{h} \lambda_i \chi_i$$

$$(f_{[g]} \mid \chi_j) = \sum_{i=1}^{h} \lambda_i (\chi_i \mid \chi_j) = \lambda_j$$

$$\Rightarrow \lambda_j = (f_{[g]} \mid \chi_j) = \frac{1}{|G|} \sum_{\in G} f_{[g]}(g')\,\overline{\chi_j(g')}$$

$$= \frac{1}{|G|} \sum_{\in C(g)} f_{[g]}(g')\,\overline{\chi_j(g')}$$

$$= \frac{c(g)}{|G|}\,\overline{\chi_j(g)}$$

$$\Rightarrow f_{[g]} = \sum_{i=1}^{h} \lambda_i \chi_i = \frac{c(g)}{|G|} \sum_{i=1}^{h} \overline{\chi_i(g)}\,\chi_i$$

(1) Then $1 = f_{[g]}(g) = \frac{c(g)}{|G|} \sum_{i=1}^{h} \overline{\chi_i(g)} \chi_i(g)$

(2) If g' is not conjugate to g,

$$0 = f_{[g]}(g') = \frac{c(g)}{|G|} \sum_{i=1}^{h} \overline{\chi_i(g)} \chi_i(g')$$

$$\Rightarrow \sum_{i=1}^{h} \overline{\chi_i(g)} \chi_i(g') = 0 \Rightarrow \sum_{i=1}^{h} \chi_i(g) \overline{\chi_i(g')} = 0$$

$\square$

$\boxed{\text{1-dim repns}}$

Let G be a finite group. Define the <u>commutator</u> of G

$$[G,G] := \langle g h g^{-1} h^{-1} : g, h \in G \rangle$$

Check: (1) $[G,G] \lhd G$

(2) $G/[G,G]$ is an abelian group.

Proposition: Let $f: G \to G'$ be a group homomorphism and G' is an abelian group. Then f factors through $[G,G]$, that is, we can find

$\tilde{f}: G/[G,G] \to G'$ such that

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G' \\ G/[G,G] \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Proof: For $g \in G$, set $\tilde{f}(g[G,G]) = f(g)$

We ~~only~~ need to show $\tilde{f}$ is well defined.

Suppose that $g_1^{-1} g_2 \in [G,G]$ $g_1^{-1} g_2 = g h g^{-1} h^{-1} \cdots$

$\tilde{f}(g_1[g_1, g_2]) = f(g_1)$ $\quad$ $\tilde{f}(g_2[G,G]) = f(g_2)$

$$f(g_1^{-1} g_2) = f(g h g^{-1} h^{-1} \cdots) = f(g) f(h) f(g)^{-1} f(h)^{-1} \cdots$$
$$= e$$

since G' is an abelian group. ∎

Remark: This is equivalent to say:

If $f: G \to G'$ and G' is abelian, then $[G,G] \subseteq \ker(f)$.

Next, let G be a group, and (π, V) an 1-dim repn of G. This is always irreducible.

Furthermore, let χ be its character. Then we can identify:

$$\pi: \quad G \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$
$$g \longmapsto \chi(g).$$

Notice that $\mathbb{C}^\times$ is an abelian group. Then

$$[G, G] \leq \operatorname{Ker}(\pi).$$

Fact: There is an one-to-one correspondence

$$\left\{ \begin{matrix} \text{1-dim repn} \\ \text{of } G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducible repn} \\ \text{of } G/[G,G] \end{matrix} \right\}$$

$G/[G,G]$ is an abelian group $\Rightarrow \# \operatorname{Irr}(G/[G,G]) = |G/[G,G]|$

$$\Rightarrow \# \text{ 1-dim repn of } G = |G/[G,G]|.$$

Example: Let $G = S_3$. Determine $\operatorname{Irr}(S_3)$.

(1) The conjugate classes of S_3

$$S_3 = \{e\} \sqcup \{(12), (13), (23)\} \sqcup \{(123), (132)\}.$$

$$\Rightarrow \# \operatorname{Irr}(S_3) = 3$$

(2) $[S_3, S_3] = \{e, (123), (132)\}$ and $S_3/[S_3, S_3] \simeq \mathbb{Z}/2\mathbb{Z}$

$$\Rightarrow \# \text{ of 1-dim repns of } S_3 = 2.$$

Proof: $[S_3, S_3] \triangleleft S_3 \Rightarrow |[S_3, S_3]| = 1, 2, 3, 6.$

S_3 is <u>not</u> an abelian group $\Rightarrow |[S_3, S_3]| = 1, 2, 3.$

$$(12)(13)(12)^{-1}(13)^{-1} = (12)(13)(12)(13) = (123)$$

$\Rightarrow (123) \in [S_3, S_3] \Rightarrow 3 \,\big|\, |[S_3, S_3]| \Rightarrow [S_3, S_3] = 3$

$\Rightarrow [S_3, S_3] = \{e, (123), (132)\}$

Then $S_3/[S_3, S_3]$ has order $2 \Rightarrow S_3/[S_3,S_3] \simeq \mathbb{Z}/2\mathbb{Z}.$

(3) Recall: $|S_3| = 6$

$$6 = \underbrace{1^2 + 1^2 + n^2}_{\#\text{ of irreducible repns}} \Rightarrow n = 2.$$

$\Rightarrow S_3$ has 2 1-dim irreducible repns

1 2-dim irreducible repns.

(4). Recall: the standard repn of S_3 $(std, \mathbb{C}^3)$

We know: $(std, \mathbb{C}^3) = (\pi_1, V_1) \oplus (\pi_2, V_2)$ both irreducible.

with $\dim V_1 = 1 \rightsquigarrow$ the trivial repn

$\dim V_2 = 3 - 1 = 2. \rightsquigarrow$ the 2-dim repn

Character Table for S_3 :

$\Rightarrow$

		e	$(12), (13), (23)$	$(123)\ (132)$
1-dim $\Leftarrow$	χ_{tri}	1	1	1
1-dim	χ_{sgn}	1	-1	1.
2-dim	χ_2	2.	0	-1.

For $V_2 = \mathrm{span}_{\mathbb{C}}\{e_1-e_2,\ e_2-e_3\}$

$$\pi_1(12)\,(e_1-e_2,\ e_2-e_3) = (e_2-e_1,\ e_1-e_3)$$
$$= (e_1-e_2,\ e_2-e_3)\begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}$$

$$\Rightarrow \chi_2(12) = -1+1 = 0$$

$$\pi(123)\,(e_1-e_2,\ e_2-e_3) = (e_2-e_3,\ e_3-e_1)$$
$$= (e_1-e_2,\ e_2-e_3)\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \chi_2(123) = -1.$$

For χ_{sgn}, we use the 2nd orthogonality relation:

$$1\cdot\chi_{tr}(12) + 1\cdot\chi_{sgn}(12) + 2\cdot\chi_2(12) = 0 \Rightarrow \chi_{sgn}(12) = -1.$$

$$1\cdot\chi_{tr}(123)+1\cdot\chi_{sgn}(123)+2\cdot\chi_2(123)=0 \Rightarrow \chi_{sgn}(123)=1.$$

Remark: We find χ_{sgn} by orthogonality relations.

However, what is sgn repn ?

Indeed:

$$\begin{aligned} sgn: S_3 &\longrightarrow \mathbb{C}^\times \\ \sigma &\longmapsto sgn(\sigma) \end{aligned}$$

$$sgn(\sigma)=\begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Useful results for finding charater table:

1st orthogonality relation:

$$\frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} = \delta_{\chi_1,\chi_2}$$

2nd orthogonality relation:

$$\sum_{i=1}^{h}\chi_i(s)\overline{\chi_i(t)} = \frac{|G|}{c(s)}\delta_{[s],[t]}$$