Let $q \geq 1$ be an integer.

Definition: An complex valued function $\chi: \mathbb{Z} \to \mathbb{C}$ is a <u>Dirichlet character modulus q</u> if

(1) (Completely multiplicative) $\chi(mn) = \chi(m)\chi(n)$ for any $m, n \in \mathbb{Z}$

(2) (Periodicity) $\chi(a+q) = \chi(a)$ for any $a \in \mathbb{Z}$

(3) $$\chi(a) = \begin{cases} 0 & \text{if } \gcd(a,q) > 1 \\ \neq 0 & \text{if } \gcd(a,q) = 1. \end{cases}$$

Example: (1) For any $q \geq 1$, we define:

$$\mathbb{1}_q(n) = \begin{cases} 1 & \text{if } (n,q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is a Dirichlet character (mod q). We call it the <u>principal</u> character.

(2) We define $$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \text{ even} \end{cases}$$

This is a Dirichlet character (mod 4).

Next, denote by $\mathbb{Z}/q\mathbb{Z} = \{\bar{0}, \cdots \overline{q-1}\}$

units in $(\mathbb{Z}/q\mathbb{Z})$

and $(\mathbb{Z}/q\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/q\mathbb{Z} : (a, q) = 1\}$

Euler totient function

$(\mathbb{Z}/q\mathbb{Z})^\times$ is an abelian group. Set $\varphi(q) = \#(\mathbb{Z}/q\mathbb{Z})^\times$

Proposition: The following statements are equivalent:

(1) G is an abelian group.

(2) All irreducible reprs of G are 1-dim.

Proof of Sketch: G is abelian $\Leftrightarrow$ $\#C(G)$ = the order of G.

Fact (Chinese Remainder Theorem): Suppose that $m = m_1 m_2$

and $(m_1, m_2) = 1$. Then:

(1) $\mathbb{Z}/m_1 m_2 \mathbb{Z} \simeq \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$

(2) $(\mathbb{Z}/m_1 m_2 \mathbb{Z})^\times \simeq (\mathbb{Z}/m_1\mathbb{Z})^\times \times (\mathbb{Z}/m_2\mathbb{Z})^\times$

Observation: There is a bijection between.

$$\left\{\begin{array}{c}\text{irreducible reprs}\\ \text{of}\\ (\mathbb{Z}/q\mathbb{Z})^\times\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{characters of}\\ \text{irreducible reprs}\\ \text{of } (\mathbb{Z}/q\mathbb{Z})^\times\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Dirichlet characters}\\ \text{modulo } q\end{array}\right\}$$

The first bijection is obvious, as $(\mathbb{Z}/q\mathbb{Z})^\times$ is an abelian group.

Next, we construct the second bijection explicitly.

Let $\tilde{\chi}$ be the character of an irreducible repn of $(\mathbb{Z}/q\mathbb{Z})^\times$

Then we define: for any $n \in \mathbb{Z}$ and $(n,q)=1$, $\bar{n} \in (\mathbb{Z}/q\mathbb{Z})^\times$

$$\chi(n) = \begin{cases} \tilde{\chi}(\bar{n}) & \text{if } (n,q)=1 \\ 0 & \text{if } (n,q)>1. \end{cases}$$

Claim: χ is a Dirichlet character $\pmod q$

(1) (Completely multiplicative) $\chi(mn) = \chi(m)\chi(n)$ if $(mn,q)=1$

$0 = \chi(mn) = \chi(m)\chi(n)$ if $(mn,q)>1$

(2) (Periodicity) $\overline{a+q} = \bar{a} \Rightarrow \chi(a+q) = \chi(a)$.

(3) $\chi(n)=0$ if $(n,q)>1$ by definition.

Conversely, Let χ be a Dirichlet character $\pmod q$.

We set
$$\begin{aligned} \tilde{\chi} : (\mathbb{Z}/q\mathbb{Z})^\times &\to \mathbb{C} \\ \bar{n} &\mapsto \chi(n) \end{aligned}$$

This is well-defined by the periodicity.

By the completely multiplicative property, $\tilde{\chi}$ is a 1-dim repn of $(\mathbb{Z}/q\mathbb{Z})^{\times}$ and we can identify it with its character.

This shows: $\boxed{\#\text{ Dirichlet characters mod } q = \varphi(q)}$

Next, let a be an integer and $(a, q) = 1$.

We define the following function:

or $q \mid (n-a)$

$$\mathbb{1}_{n \equiv a \pmod q}(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod q \\ 0 & \text{otherwise.} \end{cases}$$

This is a function with periodicity q.

We can also "restrict" $\mathbb{1}_{n \equiv a \pmod q}$ to $(\mathbb{Z}/q\mathbb{Z})^{\times}$ denoted by $\tilde{\mathbb{1}}_{n \equiv a \pmod q}$.

This automatically becomes a class function on $(\mathbb{Z}/q\mathbb{Z})^{\times}$

(Class function = function when G is abelian)

Denote by: $\{\tilde{\chi}_1, \ldots \tilde{\chi}_{\varphi(q)}\}$ the irreducible characters of $(\mathbb{Z}/q\mathbb{Z})^{\times}$. This is a basis for class functions.

$$\Rightarrow \quad \tilde{\mathbb{1}}_{n \equiv a \pmod q} = \sum_{i=1}^{\varphi(q)} \lambda_i \tilde{\chi}_i$$

with

$$\lambda_i = \left(\widetilde{\mathbb{1}}_{n\equiv a \pmod q} \,\middle|\, \widetilde{\chi}_i \right)$$

$$= \frac{1}{\varphi(q)} \sum_{g\in(\mathbb{Z}/q\mathbb{Z})^\times} \widetilde{\mathbb{1}}_{n\equiv a \pmod q}(g)\,\overline{\widetilde{\chi}_i(g)}$$

$$= \frac{1}{\varphi(q)}\,\overline{\widetilde{\chi}_i(a)}$$

$$\Rightarrow \quad \widetilde{\mathbb{1}}_{n\equiv a \pmod q} = \frac{1}{\varphi(q)} \sum_{i=1}^{\varphi(q)} \overline{\widetilde{\chi}_i(a)}\,\widetilde{\chi}_i$$

Then we "lift" it to $\mathbb{Z}$, and we conclude:

$$\boxed{\mathbb{1}_{n\equiv a \pmod q} = \frac{1}{\varphi(q)} \sum_{i=1}^{\varphi(q)} \overline{\chi_i(a)}\,\chi_i}$$

Here $\{\chi_i : 1\le i\le \varphi(q)\}$ are Dirichlet characters $\pmod q$.

Properties of Dirichlet characters.

Proposition: (1) If χ is a Dirichlet character $\pmod q$, so will $\overline{\chi}$

(2) If χ_1, χ_2 are Dirichlet characters mod q, so will $\chi_1\chi_2$.

Next, let $q_1 \mid q_2$, then we have:

$$p: \mathbb{Z}/q_2\mathbb{Z} \longrightarrow \mathbb{Z}/q_1\mathbb{Z} \qquad n \pmod{q_2} \mapsto n \pmod{q_1}$$

This implies, let $\tilde{\chi}_1$ be a Dirichlet character mod q_1, we can construct a Dirichlet character mod q_2:

$$\begin{tikzcd}
\mathbb{Z}/q_2\mathbb{Z} \arrow[r, "p"] \arrow[dr, "\tilde{\chi}_2"'] & \mathbb{Z}/q_1\mathbb{Z} \arrow[d, "\tilde{\chi}_1"] \\
 & \mathbb{C}^\times
\end{tikzcd}$$

In this case: $\chi_2 = \chi_1 \cdot \mathbb{1}_{q_2}$

Here $\mathbb{1}_{q_2}$ is the principal character mod q_2

Definition: Let $\chi \pmod q$ be a Dirichlet character.

(1) If we can find $q_1 \mid q$ $(q_1 < q)$ and $\chi' \pmod{q_1}$ such that

$$\chi = \chi' \cdot \mathbb{1}_{q_2},$$

we say χ is an <u>imprimitive</u> character.

(2) Otherwise, χ is a <u>primitive</u> character, and we call q its conductor.

Definition: Let $\chi \pmod q$ be a Dirichlet character.

(1) χ is a <u>complex</u> character if $\chi \neq \bar{\chi}$.

(2) χ is a <u>real</u> character if $\chi = \bar{\chi}$.

In this case, χ only takes value $\pm 1, 0$.

Arithmetic functions:

Definition: An arithmetic function is a function $f: \mathbb{Z} \to \mathbb{C}$.

Definition: An arithmetic function f is

(1) <u>multiplicative</u> if $f(mn) = f(m)f(n)$ when $(m,n)=1$

(2) <u>completely multiplicative</u> if $f(mn) = f(m)f(n)$ any $m,n \in \mathbb{Z}$.

Note: completely multiplicative $\Rightarrow$ multiplicative

Example: (1) $\chi \pmod q$ is a Dirichlet character $\Rightarrow$ completely multiplicative

(2) Euler totient function multiplicative

not completely multiplicative

(3) $\mathbb{1}_{n \equiv a \bmod q}(n)$ not multiplicative

not completely multiplicative.

Question: Why multiplicative funtions?

Answer: The study of multiplicative funtions can be reduced to its values at prime powers.

Let f be multiplicative, $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = f(p_1^{\alpha_1}) \cdots f(p_r^{\alpha_r})$$

Furthermore, the study of completely multiplicative funtions can be reduced to its values at primes

Let f be completely multiplicative, $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = f(p_1)^{\alpha_1} \cdots f(p_r)^{\alpha_r}.$$