Some notations

Let $f(x)$ and $g(x)$ be two functions and $g(x) \geqslant 0$.

We write $\underline{f(x) = O(g(x))}$ (or $f(x) \ll g(x)$) if

there exists a constant C such that

$$|f(x)| \leq C\, g(x).$$

We write $\underline{f(x) = o(g(x))}$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

Dirichlet's Theorem: Let a, q be integers and $(a,q) = 1$.

Then there are infinitely many primes in AP:

$$a,\ a+q,\ a+2q,\ a+3q,\ \cdots$$

Definition: Let $\chi \pmod{q}$ be a Dirichlet character.

For $s > 1$, we define <u>Dirichlet L-function</u>:

$$L(s, \chi) = \sum_{n \geqslant 1} \frac{\chi(n)}{n^s}.$$

Proposition (Prove Later):

(1) When $s > 1$, $L(s, \chi)$ is absolutely convergent

(2) When $s > 1$, we have the Euler product expansion:

$$L(s, \chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

where the (infinite) product is over all primes p.

Next, we "take" log of $L(s, \chi)$ and we expect:

$$\log L(s, \chi) = \log \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{p} \log \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

Proposition (Prove Later): $\log \frac{1}{1-z} = z + O(|z|^2)$ if $|z| < \frac{1}{2}$.

This implies:

$$\boxed{\log L(s, \chi) = \sum_{p} \frac{\chi(p)}{p^s} + O(1)}$$

Sketch of Dirichlet's Theorem:

Let p be a prime in $\{a, a+q, a+2q, \cdots\}$

Then $p \equiv a \pmod{q}$

Therefore, if we can show:

$$\lim_{s \to 1^+} \sum_{p \equiv a \pmod{q}} \frac{1}{p^s} = \infty,$$

we complete the proof of Dirichlet's Theorem.

We again introduce:

$$\mathbb{1}_{n\equiv a \pmod q}(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod q \\ 0 & \text{othewise} \end{cases}$$

and we showed:

$$\mathbb{1}_{n\equiv a \pmod q} = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\, \chi$$

Therefore:

$$\sum_{p\equiv a \pmod q} \frac{1}{p^s} = \sum_p \frac{\mathbb{1}_{n\equiv a \pmod q}(p)}{p^s}$$

$$= \sum_p \frac{1}{p^s} \cdot \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\, \chi(p)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \sum_p \frac{\chi(p)}{p^s}$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \log L(s,\chi) + O(1)$$

$$= \frac{1}{\varphi(q)} \log L(s, \mathbb{1}_q) + \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \mathbb{1}_q}} \overline{\chi(a)} \log L(s,\chi) + O(1)$$

Proposition (Prove Later):

(1) For $\mathbb{1}_q$, $\lim_{s\to 1^+} L(s,\mathbb{1}_q) = \infty \Rightarrow \lim_{s\to 1^+} \log L(s,\mathbb{1}_q) = \infty$

(2) For $\chi \neq \mathbb{1}_q$. $L(1,\chi) \neq 0 \Rightarrow \lim_{s\to 1^+} \log L(s,\chi)$ bounded.

This is the most difficult part !

This will finally shows: $\lim_{s\to 1^+} \sum_{p\equiv a \pmod q} \frac{1}{p^s} = \infty$. $\square$.

Key ingredients:

(1) The absolute convergence of an infinite series/product.

(2) What is the log-function for complex numbers?

(3) How to show $L(1,\chi) \neq 0$ when $\chi \neq \mathbb{1}_q$.

We will discuss (1) and (2) in Part II

and (3) in Part III.

Infinite series

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Denote by

$$S_n = \sum_{k=1}^{n} a_k \quad \text{its } n\text{-th pertial sum.}$$

Definition: An infinite series is <u>convergent</u> if $\lim_{n\to\infty} S_n$ exists

Definition: An infinite series is <u>absolutely convergent</u> if

$\sum_{n=1}^{\infty} |a_n|$ is convergent. (or $\sum_{n=1}^{\infty} |a_n| < \infty$)

We can generalize this to a series of funtions:

Let $\sum_{n=1}^{\infty} f_n(x)$ be an infinite series of funtions. domain I.

Definition: $\sum_{n=1}^{\infty} f_n(x)$ is <u>convergent</u> if for each $x \in I$.

its n-th paritial is convergent.

Definition: $\sum_{n=1}^{\infty} f_n(x)$ is <u>absolutely convergent</u> if

$\sum_{n=1}^{\infty} |f_n(x)|$ is convergent for all $x \in I$

$$\left(\sum_{n=1}^{\infty} |f_n(x)| < \infty \right)$$

Example: $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

When $s>1$, $L(s,\chi)$ is absolutely convergent.

Proof: It suffices to show: $\sum_{n=1}^{\infty} \left|\frac{\chi(n)}{n^s}\right| < \infty$ for any $s>1$.

Notice that $|\chi(n)| \in \{1,0\} \Rightarrow \left|\frac{\chi(n)}{n^s}\right| \leq \frac{1}{n^s}$

By p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent when $s>1$

Therefore, $L(s,\chi)$ is absolutely convergent when $s>1$. $\square$.

Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (= L(s, \mathbb{1}_1))$

Proposition: (1) $\lim_{s\to 1^+} \zeta(s) = \infty$

(2) For $1<s<2$, $\zeta(s) = O\left(\frac{1}{s-1}\right)$

Proof: Observation: $\int_n^{n+1} \frac{1}{x^s}\,dx \leq \frac{1}{n^s} \leq \int_{n-1}^{n} \frac{1}{x^s}\,dx$

only prove $\int_n^{n+1} \frac{1}{x^s}\,dx \leq \frac{1}{n^s}$:

when $n \leq x \leq n+1$, $\frac{1}{x^s} \leq \frac{1}{n^s}$

This shows: $\int_n^{n+1} \frac{1}{x^s}\,dx \leq \int_n^{n+1} \frac{1}{n^s}\,dx = \frac{1}{n^s}(n+1-n)$
$= \frac{1}{n^s}$.

(1) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^s}\,dx$

$= \int_1^{\infty} \frac{1}{x^s}\,dx = \left. \frac{x^{1-s}}{1-s} \right|_1^{\infty} = \frac{1}{s-1}$.

$\Rightarrow \lim_{s \to 1^+} \zeta(s) \geq \lim_{s \to 1^+} \frac{1}{s-1} = \infty$

(2) $\zeta(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^s} \leq 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{1}{x^s}\,dx$

$= 1 + \int_1^{\infty} \frac{1}{x^s}\,dx = 1 + \frac{1}{s-1}$.

When $s < 2$, $1 < \frac{1}{s-1}$

$\zeta(s) \leq 1 + \frac{1}{s-1} < \frac{1}{s-1} + \frac{1}{s-1} = \frac{2}{s-1}$.

$\Rightarrow \zeta(s) = O\left(\frac{1}{s-1}\right)$ $\square$

Infinite produt.

Let $\{a_n\}$ be a sequence, and we want to study the infinite prodult: $\prod_{n=1}^{\infty} a_n$

Set $P_n = \prod_{k=1}^{n} a_k$ n-th partial product

Definition: An infinite product is <u>convergent</u> if

$\lim_{n\to\infty} P_n$ exists.

<u>Proposition</u>: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and $a_n \neq 1$ for all n. Then the infinite product:

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-a_n} \right)$$

is convergent. Moreover, the product is non-zero.