For $|z|<1$, we define a differentiable function:

$$\log_1\left(\frac{1}{1-z}\right) := \sum_{n=1}^{\infty} \frac{z^n}{n}$$

Proposition: (1) If $|z|<1$, then $e^{\log_1 \frac{1}{1-z}} = \frac{1}{1-z}$.

(2) If $|z_1|, |z_2| < \frac{1}{2}$, then

$$\log_1\left(\frac{1}{1-z_1}\right)\left(\frac{1}{1-z_2}\right) = \log_1 \frac{1}{1-z_1} + \log_2 \frac{1}{1-z_2}$$

(3) If $|z|<1$, then

$$\log_1 \frac{1}{1-z} = z + E_1(z)$$

When $|z|<\frac{1}{2}$, $E_1(z) = O(|z|^2)$

(4) If $|z|<\frac{1}{2}$, then

$$\left|\log_1 \frac{1}{1-z}\right| \leqslant 2|z| \quad \left(\log_1 \frac{1}{1-z} = O(|z|)\right)$$

Proof: (1) It suffices to show: when $|z|<1$

$$(1-z)\, e^{\log_1 \frac{1}{1-z}} = 1.$$

Set $z = re^{i\theta}$ then $0 \leq r < 1$, $\theta \in [0, 2\pi)$.

$$F(r,\theta) = (1-re^{i\theta})\, e^{\log_1 \frac{1}{1-re^{i\theta}}}$$

We show: $\frac{\partial F}{\partial r} \equiv \frac{\partial F}{\partial \theta} \equiv 0$ for $0 \le r < 1$, $\theta \in [0, 2\pi)$.

(Only show $\frac{\partial F}{\partial r} \equiv 0$):

$$\frac{\partial F}{\partial r} = (-e^{i\theta})\, e^{\log_1 \frac{1}{1-re^{i\theta}}} + (1-re^{i\theta})\left(e^{\log_1 \frac{1}{1-re^{i\theta}}}\right)'$$

$$= \Big\{\underbrace{\Big(-e^{i\theta} + (1-re^{i\theta})\frac{\partial}{\partial r}\Big(\log_1 \frac{1}{1-re^{i\theta}}\Big)}_{g(r,\theta)}\Big\}\, e^{\log_1 \frac{1}{1-re^{i\theta}}}$$

We only need to show $g(r,\theta) \equiv 0$.

$$\frac{\partial}{\partial r}\left(\log_1 \frac{1}{1-re^{i\theta}}\right) = \frac{\partial}{\partial r}\left(\sum_{n=1}^{\infty} \frac{(re^{i\theta})^n}{n}\right) = \frac{\partial}{\partial r}\left(\sum_{n=1}^{\infty} \frac{r^n e^{in\theta}}{n}\right)$$

$$= \sum_{n=1}^{\infty} r^{n-1} e^{in\theta} = e^{i\theta} \cdot \sum_{n=1}^{\infty} (re^{i\theta})^{n-1}$$

$$= e^{i\theta} \cdot \frac{1}{1-re^{i\theta}} \quad (\text{since } |re^{i\theta}| < 1)$$

$$\Rightarrow g(r,\theta) = -e^{i\theta} + e^{i\theta} \cdot \frac{1}{1-re^{i\theta}} \cdot (1-re^{i\theta})$$

$$= -e^{i\theta} + e^{i\theta} = 0.$$

A similar way will show: $\frac{\partial F}{\partial \theta} \equiv 0$

Sime $F(r,\theta)$ is a differentiable funtion,

$F(r,\theta) \equiv C$ a constant

$$F(0,0) = (1 - 0\cdot e^{i\cdot 0})\, e^{\log_1 \frac{1}{1-0}} = 1.$$

$$\Rightarrow F(r,\theta) \equiv 1.$$

(2) Check: when $|z_1|, |z_2| < 1$, $\frac{1}{1-z_1}\cdot\frac{1}{1-z_2} = \frac{1}{1-w}$ with $|w| < 1$

By (1), $$e^{\log_1\left(\frac{1}{1-z_1}\cdot\frac{1}{1-z_2}\right)} = \frac{1}{1-z_1}\cdot\frac{1}{1-z_2}$$

$$= e^{\log_1 \frac{1}{1-z_1}}\, e^{\log_1 \frac{1}{1-z_2}} = e^{\log_1 \frac{1}{1-z_1} + \log_2 \frac{1}{1-z_2}}.$$

$$\Rightarrow \log_1\left(\frac{1}{1-z_1}\cdot\frac{1}{1-z_2}\right) = \log_1 \frac{1}{1-z_1} + \log_2 \frac{1}{1-z_2} + 2\pi i\cdot M(z_1,z_2)$$

Here $M(z_1,z_2) \in \mathbb{Z}$ and $M(z_1,z_2)$ is a continuous function.

This will force $M(z_1,z_2) \equiv C$. (Intermediate Value Theorem)

Take $z_1 = z_2 = 0 \Rightarrow M(0,0) = 0 \Rightarrow M(z_1,z_2) \equiv 0.$

This proves. (2).

(3) $$\log_1 \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n} = z + \sum_{n\geq 2} \frac{z^n}{n}$$

Set $$E_1(z) = \sum_{n=2}^{\infty} \frac{z^n}{n}.$$

When $|z| < \frac{1}{2}$,

$$|E_1(z)| \leq \sum_{n=2}^{\infty} \left|\frac{z^n}{n}\right| = \sum_{n=2}^{\infty} \frac{|z|^n}{n}$$

$$\leq \sum_{n\geq 2}^{\infty} |z|^n = \frac{|z|^2}{1-|z|} \leq 2|z|^2 \quad (|z|<\tfrac{1}{2})$$

Therefoe, $E_1(z) = O(|z|^2)$ when $|z|<\frac{1}{2}$.

(4) $$\left|\log_1 \frac{1}{1-z}\right| = \left|\sum_{n=1}^{\infty} \frac{z^n}{n}\right| \leq \sum_{n=1}^{\infty} \left|\frac{z^n}{n}\right| = \sum_{n=1}^{\infty} \frac{|z|^n}{n}$$

$$\leq \sum_{n=1}^{\infty} |z|^n \leq \frac{|z|}{1-|z|} \leq 2|z| \quad (|z|<\tfrac{1}{2}).$$ □

<u>Proposition:</u> If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and $a_n \neq 1$ for all n. Then the infinite product:

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-a_n}\right)$$

is convergent. Moreover, the produt is non-zero.

Proof: Sime $\sum_{n=1}^{\infty} a_n$ is absolutely convergent,

$\Rightarrow \lim_{n\to\infty} |a_n| = 0$

$\Rightarrow$ except for finitely many terms, $|a_n| \leq \frac{1}{2}$.

Without loss of generality, we can assume $|a_n| \leq \frac{1}{2}$ for all n.

$$\prod_{n=1}^{N}\left(\frac{1}{1-a_n}\right) = \prod_{n=1}^{N} e^{\log_1 \frac{1}{1-a_n}} = e^{\sum_{n=1}^{N} \log_1 \frac{1}{1-a_n}}$$

By (4), $\left|\sum_{n=1}^{N} \log_1 \frac{1}{1-a_n}\right| \leq \sum_{n=1}^{N} 2|a_n| = 2\sum_{n=1}^{N} |a_n|$

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Rightarrow \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| < \infty$

This implies: $\lim_{N\to\infty} \sum_{n=1}^{N} \log_1 \frac{1}{1-a_n} = A$.

Therefore, $\lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1-a_n} = e^A$ and

$$\prod_{n=1}^{\infty} \frac{1}{1-a_n} = e^A \neq 0. \qquad \#$$

For $\chi \pmod q$, a Dirichlet character and $s>1$,

Dirichlet L-function: $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

We prove the following result for $L(s,\chi)$.

Theorem: The Dirichlet L-function has a Euler product:

$$L(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1-\frac{\chi(p)}{p^s}} \neq 0$$

when $s>1$.

Proof: When $s>1$, $L(s,\chi)$ is absolutely convergent.

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \qquad n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$$

χ is multiplicative

$$=\sum_{n=1}^{\infty}\frac{\chi(p_1^{\alpha_1})}{p_1^{\alpha_1 s}}\cdots\frac{\chi(p_r^{\alpha_r})}{p_r^{\alpha_r s}}$$

$$=\prod_{p \text{ prime}}\left(1+\frac{\chi(p)}{p^s}+\frac{\chi(p^2)}{p^{2s}}+\cdots\right)$$

χ is completely multiplicative

$$=\prod_{p \text{ prime}}\left(1+\frac{\chi(p)}{p^s}+\left(\frac{\chi(p)}{p^s}\right)^2+\left(\frac{\chi(p)}{p^s}\right)^3+\cdots\right)$$

$$=\prod_{p \text{ prime}}\frac{1}{1-\frac{\chi(p)}{p^s}}.$$

Finally, we need to make sure that $\prod_{p \text{ prime}}\frac{1}{1-\frac{\chi(p)}{p^s}}$ is convergent and nonzero when $s>1$.

Fix $s>1$, set $a_n=\begin{cases}\frac{\chi(p)}{p^s} & n=p\text{, a prime}\\ 0 & \text{otherwise.}\end{cases}$

Then $$\prod_{n=1}^{\infty}\frac{1}{1-a_n}=\prod_{p \text{ prime}}\frac{1}{1-\frac{\chi(p)}{p^s}}$$

Obviously, $a_n \neq 1$ for all n, and

$$\sum_{n=1}^{\infty} |a_n| = \sum_{p \text{ prime}} \left| \frac{\chi(p)}{p^s} \right| \leq \sum_{p \text{ prime}} \frac{1}{p^s} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty$$

This is absolutely convergent and we can apply the proposition. $\square$

Exercise: Let $\varphi(n)$ be the Euler totient function.

Show that, when $s > 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$

Corollary: (1) Let $\mathbb{1}_q$ be the principal character mod q.

Then $$L(s, \mathbb{1}_q) = \zeta(s) \cdot \prod_{\substack{p \mid q \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)$$

This shows: as $s \to 1^+$, $L(s, \mathbb{1}_q) \to \infty$.

(2) Let $\chi \pmod q$ be an imprimitive character.

Assume that $\chi = \chi_1 \cdot \mathbb{1}_q$ where $\chi_1 \pmod{q_1}$. Then

$$L(s, \chi) = L(s, \chi_1) \cdot \prod_{\substack{p \mid q \\ p \nmid q_1}} \left(1 - \frac{\chi_1(p)}{p^s}\right)$$

p prime

Proof: (1) is a special case of (2). We only prove (2):

$$L(s,\chi_1) = \prod_{p} \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1} = \prod_{p \nmid q_1} \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}$$

$(\chi_1(p) = 0 \text{ if } p \mid q_1)$

$$= \prod_{p \nmid q} \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1} \cdot \prod_{\substack{p \nmid q_1 \\ p \mid q}} \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}$$

When $p \nmid q$, $\chi_1(p) = \chi(p)$ (since $\chi = \chi_1 \cdot \mathbb{1}_q$)

Notice: $L(s,\chi) = \prod_{p \nmid q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$

$$= L(s,\chi) \cdot \prod_{\substack{p \mid q \\ p \nmid q_1}} \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}$$

$$\Rightarrow L(s,\chi) = L(s,\chi_1) \cdot \prod_{\substack{p \mid q \\ p \nmid q_1}} \left(1 - \frac{\chi_1(p)}{p^s}\right) \qquad \square$$