Definition: Let $f: I \subseteq \mathbb{R} \to \mathbb{C}$ be a funtion. Let $k \geq 1$ be an integer. We say $\underline{f \in C^k(I)}$ if for any $x \in I$, f is k-th diffentiable and $f^{(k)}(x)$ is continuous.

f is <u>smooth</u> if $f \in C^\infty(I) = \prod_{k \geq 1} C^k(I)$

Let $I \subseteq \mathbb{R}$ be an interval. Let $\{f_n(x)\}$ and $\{g_n(x)\}$ be two sequences of funtions: $f_n : I \to \mathbb{C}$, $g_n : I \to \mathbb{R}$.

Question: Under what conditions, $\sum_{n=1}^{\infty} f_n(x)\, g_n(x)$ is convergent?

<u>Ans</u>: Possible method: Dirichlet test.

Fact: (Dirichlet test). Suppose that $\{f_n(x)\}$, $\{g_n(x)\}$ satisfy the following conditions:

(1) $F_N(x) = \sum_{n=1}^{N} f_n(x)$, $|F_N(x)| \leq M$ and M is independent from n, x

(2) For each fixed $x \in I$, $\cdots g_{n+1}(x) \leq g_n(x) \leq g_{n-1}(x) \cdots$ and $\lim_{n\to\infty} g_n(x) = 0$

(3) each $g_n(x)$ is a monotone funtion (increasing/decreasing)

Then $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ is convergent for $x \in I$. Moreover;

① If all $f_n(x), g_n(x)$ are continuous, then

$$\sum_{n=1}^{\infty} f_n(x) g_n(x) \quad \text{is continuous}$$

② If all $f_n(x), g_n(x)$ are differentiable, then

$$\sum_{n=1}^{\infty} f_n(x) g_n(x) \quad \text{is differentiable., and}$$

$$\left(\sum_{n=1}^{\infty} f_n(x) g_n(x) \right)' = \sum_{n=1}^{\infty} f_n'(x) g_n(x) + f_n(x) g_n'(x)$$

Using Dirichlet's test, we can study $L(s, \chi)$ when χ is not a principal character.

<u>Proposition</u>: Let χ be a nonprincipal character mod q.

(1) $L(s,\chi) \in C^1((0,\infty))$, that is, for $s \in (0,\infty)$, $L(s,\chi)$ is differentiable and its derivative $L'(s,\chi)$ is continuous.

(2) As $s \to \infty$, $L(s,\chi) = 1 + O(2^{-s})$

As $s \to \infty$ $L'(s,\chi) = O((\sqrt{2})^{-s})$

$\boxed{\text{Notice}}$: (2) is also true when $\chi = \mathbb{1}_q$, the principal character.

Idea of proof: Set $I = [0, \infty)$

$$f_n(s) = \chi(n) \quad \left(F_N(s) = \sum_{n=1}^{N} \chi(n) \right) \qquad g_n(s) = \frac{1}{n^s}$$

$f_n(s)$ are constant functions $\quad g_n(s)$ are differentiable functions.

Then we want to apply Dirichlet's test.

Lemma: Let $\chi \pmod q$ be a non principal character.

Then
$$\left| \sum_{n=1}^{N} \chi(n) \right| \leq q.$$

for any $N \geq 1$.

Proof: Recall: for $(\mathbb{Z}/q\mathbb{Z})^\times$, we have the orthogonality relation:

$$\sum_{g \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi_1(g) \overline{\chi_2(g)} = \begin{cases} \varphi(q) & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for any two different Dirichlet characters $\pmod q$

$$\sum_{\substack{1 \leq n \leq q \\ (n,q)=1}} \chi_1(n) \overline{\chi_2(n)} = 0$$

Take $\chi_1 = \chi$, $\chi_2 = \mathbb{1}_q$ $\Rightarrow$ $\displaystyle\sum_{\substack{1 \leq n \leq q \\ (n,q)=1}} \chi(n) = 0$

Futhemore, if $(n,q)>1$, $\chi(n)=0$ This implies:

$$\sum_{1\le n\le q} \chi(n)=0.$$

Next, for any $N\in\mathbb{Z}$, $N=aq+b$ for $1\le b\le q$.

Then:

$$\sum_{n=1}^{N}\chi(n)=\sum_{n=1}^{aq}\chi(n)+\sum_{n=aq+1}^{aq+b}\chi(n)$$

$$=\sum_{n=1}^{q-1}\chi(n)+\sum_{n=q+1}^{2q}\chi(n)+\cdots+\sum_{n=(a-1)q+1}^{aq}\chi(n)$$

$$+\sum_{m=1}^{b}\chi(m+aq)$$

$$=0+\cdots 0+\sum_{m=1}^{b}\chi(m)$$

$$\Rightarrow \left|\sum_{n=1}^{N}\chi(n)\right| \le \left|\sum_{m=1}^{b}\chi(m)\right| \le \sum_{m=1}^{b}|\chi(m)| \le b\le q \quad \square$$

Proof of Proposition:

(1) We check the conditions for Dirichlet's Test.

① $\left|\sum_{n=1}^{N} F_n(s)\right| = \left|\sum_{n=1}^{N}\chi(n)\right| \le q$ independent of N, s.

② Fix $s>0$, $\cdots\le \frac{1}{(n+1)^s}\le\frac{1}{n^s}\le\frac{1}{(n-1)^s}\cdots$

and $\lim_{n\to\infty} g_n(s) = \lim_{n\to\infty} \frac{1}{n^s} = 0$

③. $g_n(s) = \frac{1}{n^s}$ This is decreasing.

Therefore, $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is differentiable when $s>0$.

and $L'(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n^s}$

Next, set $f_n(s) = \chi(n)$ $g_n(x) = \frac{\log n}{n^s}$

Then we apply Dirichlet's test again and we show:

$L'(s,\chi)$ is continuous for $s>1$.

(2) We consider $s \to \infty$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}$$

Only show: $\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} = O(2^{-s})$

$$\left| \sum_{n=2}^{\infty} \frac{\chi(n)}{n^s} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

$$\leq \frac{1}{2^s}\left(1 + 1 + \frac{1}{2^s} + \frac{1}{2^s} + \cdots\right)$$

$$\leq \frac{2}{2^s}\zeta(s) \leq \frac{4}{2^s}$$ when $s \to \infty$.

$$L'(s,\chi) = \sum_{n=2}^{\infty} \frac{\chi(n)\log n}{n^s} \qquad \log n \le A \cdot n^{\frac{s}{2}}$$

$$\Rightarrow \left| L'(s,\chi) \right| \le \sum_{n=2}^{\infty} \frac{\log n}{n^s} \le A \sum_{n=2}^{\infty} \frac{1}{n^{\frac{s}{2}}}$$

$$\le A \frac{2}{2^{\frac{s}{2}}} \zeta\left(\frac{s}{2}\right) = O\left(\frac{1}{(\sqrt{2})^s}\right) \quad \text{as } s \to \infty$$

Definition: A funtion $f(x)$ is called <u>exponential decay</u> if

for some $c > 1$. $f(s) = O\left(\frac{1}{c^s}\right)$ as $s \to \infty$

Observation: If we have an exponential decay funtion, by the comparison test, we can define

$$G(s) = -\int_s^{\infty} f(t)\,dt. \quad \text{and} \quad G'(s) = f(s)$$

when $f(s)$ is always defined.

By the proposition: for $\chi \pmod q$, when $s > 1$,

$$L(s,\chi) = 1 + O\left(\frac{1}{2^s}\right) \qquad s \to \infty$$

$$L'(s,\chi) = O\left(\frac{1}{(\sqrt{2})^s}\right) \qquad s \to \infty$$

$$\Rightarrow \frac{L'(s,\chi)}{L(s,\chi)} = O\left(\frac{1}{(\sqrt{2})^s}\right) \qquad \text{as} \to \infty$$

and when $s > 1$, $L(s,\chi) \neq 0$

Therefore, we define the 2nd log function

$$\log_2 L(s,\chi) = -\int_s^{\infty} \frac{L'(t,\chi)}{L(t,\chi)}\,dt.$$

Then $\left(\log_2 L(s,\chi)\right)' = \dfrac{L'(s,\chi)}{L(s,\chi)}$

Proposition: If $s>1$, then

(1) $e^{\log_2 L(s,\chi)} = L(s,\chi)$

(2) $\log_2 L(s,\chi) = \sum_p \log_1 \dfrac{1}{1-\frac{\chi(p)}{p^s}}$

Proof: (1) Set $g(s) = e^{-\log_2 L(s,\chi)} \cdot L(s,\chi)$

$$g'(s) = e^{\log_2 L(s,\chi)} \left\{ \left(-\log_2 L(s,\chi)\right)' \cdot L(s,\chi) + L'(s,\chi) \right\}$$

$$= e^{\log_2 L(s,\chi)} \left(-\frac{L'(s,\chi)}{L(s,\chi)} \cdot L(s,\chi) + L'(s,\chi) \right)$$

$$= 0$$

$\Rightarrow \quad e^{-\log_2 L(s,\chi)} \cdot L(s,\chi) = C$

Take $s \to \infty$ $\qquad \lim\limits_{s\to\infty} -\log_2 L(s,\chi) = \lim\limits_{s\to\infty} \int_s^{\infty} \dfrac{L'(t,\chi)}{L(t,\chi)}\,dt = 0$

$\qquad \lim\limits_{s\to\infty} L(s,\chi) = 1$

$$\lim_{s\to\infty} e^{-\log_2 L(s,\chi)} \cdot L(s,\chi) = e^0 \cdot 1 = 1 = C$$

(2) By (1) $e^{\log_2 L(s,\chi)} = L(s,\chi)$

On the other hand,

$$e^{\sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}}} = \prod_p e^{\log_1 \frac{1}{1-\frac{\chi(p)}{p^s}}} = \prod_p \frac{1}{1-\frac{\chi(p)}{p^s}} = L(s,\chi)$$

$$\Rightarrow e^{\log_2 L(s,\chi)} = e^{\sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}}}$$

Recall: $e^z = e^{z'} \Rightarrow z = z' + 2k\pi i \quad k \in \mathbb{Z}$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} + 2\pi i \cdot M(s)$$

$M(s)$ is a continuous funtion but $M(s) \in \mathbb{Z}$

This will force $M(s)$ to be a constant funtion: $M(s) \equiv C$

(Intermediate Value Theorem)

Again, take $s \to \infty$, $M(s) = 0$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} \qquad \blacksquare.$$

Corollary: When $s > 1$,

$$\sum_p \frac{\chi(p)}{p^s} = \log_2 L(s,\chi) + O(1).$$

Proof: We know $\log_2 L(s,\chi) = \sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}}$.

When $s>1$ and p a prime, $\left|\frac{\chi(p)}{p^s}\right| \leq \frac{1}{2}$

$$\Rightarrow \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} = \frac{\chi(p)}{p^s} + O\left(\frac{1}{p^{2s}}\right)$$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + O\left(\sum_p \frac{1}{p^{2s}}\right)$$

$$\sum_p \frac{1}{p^{2s}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + O(1)$$

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