Corollary: When $s>1$,

$$\sum_p \frac{\chi(p)}{p^s} = \log_2 L(s,\chi) + O(1).$$

Proof: We know $\log_2 L(s,\chi) = \sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}}$.

When $s>1$ and p a prime, $\left|\frac{\chi(p)}{p^s}\right| \le \frac{1}{2}$

$$\Rightarrow \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} = \frac{\chi(p)}{p^s} + O\left(\frac{1}{p^{2s}}\right)$$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + O\left(\sum_p \frac{1}{p^{2s}}\right)$$

$$\sum_p \frac{1}{p^{2s}} \le \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \log_2 L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + O(1)$$

$\square$

Proposition: Let $\mathbb{1}_q$ be the principal character mod q.

Then $\lim_{s\to 1^+} \log_2 L(s,\mathbb{1}_q) = \infty$.

Proof: Observation: for $L(s,\mathbb{1}_q)$,

$$\log_2 L(s,\mathbb{1}_q) = \log L(s,\mathbb{1}_q)$$

Here "log" is the usual log function.

(Idea of proof: $\left(\log L(s, \mathbb{1}_q)\right)' = \dfrac{L'(s, \mathbb{1}_q)}{L(s, \mathbb{1}_q)}$.)

Therefore, $\lim_{s \to 1^+} \log_2 L(s, \mathbb{1}_q) = \lim_{s \to 1^+} \log L(s, \mathbb{1}_q)$

$$= \log \lim_{s \to 1^+} L(s, \mathbb{1}_q) = \infty$$

(Recall: $L(s, \mathbb{1}_q) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$ and $\lim_{s \to 1^+} \zeta(s) = \infty$) $\square$

Proposition: Let $\chi \pmod q$ be a nonprincipal character.

Suppose that $L(1, \chi) \neq 0$. Then we can find a constant C

$$\left| \log_2 L(s, \chi) \right| \leq C$$

when $s \in (1, 2)$

Fact: Let $f : I \to \mathbb{C}$ be a continous funtion

Suppose that I is a closed interval and $f(x) \neq 0$ for any $x \in I$. Then we can find $A > 0$ such that

$$|f(x)| \geq A \quad \text{for any } x \in I.$$

Proof of proposition: Assume that $L(1, \chi) \neq 0$.

Recall: $\lim_{s\to\infty} L(s,\chi) = 1$.

Then we can find a large M, such that

$$\left|L(s,\chi)\right| \geq \frac{1}{2} \quad \text{when} \quad s \geq M.$$

Next, $L(1,\chi) \neq 0 \Rightarrow L(s,\chi) \neq 0$ for $s \in [1, M]$

Then $\left|L(s,\chi)\right| \geq A$ when $s \in [1, M]$

This implies: $\left|L(s,\chi)\right| \geq C'$ when $s \in [1, \infty)$

Then: for $s \in (1,2)$

$$\left|\log_2 L(s,\chi)\right| = \left|\int_s^\infty \frac{L'(s,\chi)}{L(s,\chi)}\, ds\right|$$

$$\leq \int_s^\infty \left|\frac{L'(s,\chi)}{L(s,\chi)}\right| ds$$

$$\leq \frac{1}{C'}\int_s^\infty \left|L'(s,\chi)\right| ds$$

Since $L'(s,\chi) = O\left(\frac{1}{(\sqrt{2})^s}\right)$, we can show:

$\left|\log_2 L(s,\chi)\right| \leq C$ for some constant C. $\square$

Theorem: Let a, q be integers and $(a,q)=1$.

Suppose that for any $\chi \pmod q$, a non principal character,

$$L(1,\chi) \neq 0$$

Then there are ∞-many primes in the arithmetic progression:

$$a,\ a+q,\ a+2q,\ a+3q,\ \cdots$$

Proof: It suffices to show:

$$\lim_{s\to 1^+} \sum_{p\equiv a \pmod q} \frac{1}{p^s} = \infty$$

$$\sum_{p\equiv a \pmod q} \frac{1}{p^s} = \sum_{p \text{ prime}} \frac{\mathbb{1}_{n\equiv a \pmod q}(p)}{p^s}$$

$$= \sum_{p \text{ prime}} \frac{1}{p^s} \cdot \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\ \chi(p)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \sum_{p \text{ prime}} \frac{\chi(p)}{p^s}$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \left(\log_2 L(s,\chi) + O(1) \right)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \log_2 L(s,\chi) + O(1)$$

$$\text{RHS} = \frac{1}{\varphi(q)} \log_2 L(s, \mathbb{1}_q) + \frac{1}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \mathbb{1}_q}} \overline{\chi(a)} \log_2 L(s, \chi) + O(1)$$

We showed: (1) $\lim_{s \to 1^+} \log_2 L(s, \mathbb{1}_q) = \infty$

(2) If $L(1, \chi) \neq 0$, then $|\log_2 L(s, \chi)| \leq C$ when $s \in (1, 2)$

This implies: $\lim_{s \to 1^+} \text{RHS} = \infty$, that is

$$\lim_{s \to 1^+} \sum_{p \equiv a \,(\mathrm{mod}\, q)} \frac{1}{p^s} = \infty \qquad \square$$

<u>Final goal:</u> Let $\chi \,(\mathrm{mod}\, q)$ be a non principal character.

Then $L(1, \chi) \neq 0$.

Recall: χ is a <u>complex character</u> if $\chi \neq \overline{\chi}$

χ is a <u>real character</u> if $\chi = \overline{\chi}$.

Proposition: Let $\chi \,(\mathrm{mod}\, q)$ be a complex character.

Then $L(1, \chi) \neq 0$.

Lemma: When $s > 1$

$$\prod_{\chi \pmod q} L(s,\chi) \geqslant 1.$$

Proof: $$\prod_{\chi \pmod q} L(s,\chi) = \prod_{\chi \pmod q} \exp\left(\sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} \right)$$

$$= \exp\left(\sum_{\chi \pmod q} \sum_p \log_1 \frac{1}{1-\frac{\chi(p)}{p^s}} \right)$$

$$= \exp\left(\sum_{\chi \pmod q} \sum_p \sum_{\ell \geqslant 0.} \left(\frac{\chi(p)}{p^s}\right)^\ell \cdot \frac{1}{\ell} \right)$$

$$= \exp\left(\sum_{\chi \pmod q} \sum_p \sum_{\ell \geqslant 0} \frac{\chi(p^\ell)}{\ell p^{\ell s}} \right)$$

$$= \exp\left(\sum_p \sum_{\ell \geqslant 0} \frac{1}{\ell p^{\ell s}} \cdot \sum_{\chi \pmod q} \chi(p^\ell) \right)$$

Exercise: $$\sum_{\chi \pmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \pmod q \\ 0 & \text{otherwise.} \end{cases}$$

This implies: $$\sum_p \sum_{\ell \geqslant 0} \frac{1}{\ell p^{\ell s}} \sum_{\chi \pmod q} \chi(p^\ell) \geqslant 0$$

$$\Rightarrow \prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi) \geqslant e^0 = 1. \qquad \square$$

Proof of Proposition: Proof by contradiction.

Suppose that χ_0 is a complex character and $L(1,\chi_0)=0$.

By Lagrange's mean value theorem: $s \in (1,2)$

$$L(s,\chi_0) - L(1,\chi_0) = L'(\xi,\chi_0)(s-1)$$

$$\Rightarrow |L(s,\chi_0)| \leqslant A \cdot |s-1|$$

Since χ is complex, $L(1,\chi_0)=0 \Rightarrow L(1,\overline{\chi_0})=0$

$$\Rightarrow |L(s,\overline{\chi_0})| \leqslant A' |s-1|$$

Then for $s \in (1,2)$

$$\left| \prod_{\chi} L(s,\chi) \right|$$

$$\leqslant |L(s,\mathbb{1}_q)| \cdot |L(s,\chi_0)| \cdot |L(s,\overline{\chi_0})| \cdot \prod_{\substack{\chi \neq \mathbb{1}_q \\ \chi_0, \overline{\chi_0}}} |L(s,\chi)|$$

$$\leqslant \frac{M}{|s-1|} \cdot A|s-1| \cdot A'|s-1| \cdot C$$

$$= O(|s-1|)$$

This yields: $\lim_{s \to 1} \prod_{\chi \pmod q} L(s,\chi) = 0$

A contradiction to the lemma! $\square$