Recall: let $\chi \pmod q$ be a real Dirichlet character and $\chi \neq \mathbb{1}_q$

If $L(1, \chi) \neq 0$, then Dirichlet's Theorem is valid.

In this class, we will introduce some fundamental results for real Dirichlet characters.

In this class, we always assume that χ is nonprincipal.

Observation: suppose that $\chi \pmod q$ is an imprimitive character. Then we can find a <u>primitive</u> character $\chi' \pmod{q'}$ s.t.

$$q' \mid q \quad \text{and} \quad \chi = \chi' \cdot \mathbb{1}_q.$$

In this case, we have:

$$L(s, \chi) = L(s, \chi') \cdot \prod_{\substack{p \nmid q' \\ p \mid q}} \left(1 - \frac{\chi'(p)}{p^s}\right)$$

This implies:

$L(1, \chi) \neq 0$ if and only if $L(1, \chi') \neq 0$

Therefore, our goal becomes:

for any real primitive character χ (nonprincipal), $L(1, \chi) \neq 0$.

Question: How to construct real primitive characters?

Ans: Using the quadratic symbols!

Definition: Let n, q be two integers. We say:

(1) n is a quadratic residue mod q (QR) if

$$x^2 \equiv n \pmod q \text{ has a solution in } (\mathbb{Z}/q\mathbb{Z})^\times$$

(2) n is a quadratic nonresidue mod q (NR) if

$$x^2 \equiv n \pmod q \text{ has no solution in } (\mathbb{Z}/q\mathbb{Z})^\times$$

Fact: Fix the modulus p, an odd prime, in $(\mathbb{Z}/p\mathbb{Z})^\times$

$$QR \cdot QR = QR, \quad QR \cdot NR = NR, \quad NR \cdot NR = QR.$$

Observation: QR behaves like "$+1$"

NR behaves like "-1"

Definition: Let p be an odd prime. We define the Legendre symbol: for $n \in \mathbb{Z}$

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a QR.} \\ -1 & \text{if } n \text{ is a NR.} \\ 0 & \text{if } (n,p) > 1. \end{cases}$$

The Fact above implies:

$$\left(\frac{mn}{p}\right) = \left(\frac{m}{p}\right) \cdot \left(\frac{n}{p}\right) \quad \text{for any } m, n \in \mathbb{Z}.$$

Theorem (Quadratic Reciprocity Law) Let p be an odd prime.

(1) $$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

(2) $$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod 8 \\ -1 & \text{if } p \equiv 3, 5 \pmod 8 \end{cases}$$

(3) Let p, q be two odd primes,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$$

$\left(\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = -1 \text{ iff } p \equiv 3 \pmod 4 \text{ and } q \equiv 3 \pmod 4 \right).$

This can help to calculate $\left(\frac{n}{p}\right)$ for any $n \in \mathbb{Z}$.

Finally, it is easy to check, for any $n \in \mathbb{Z}$

$$\left(\frac{n+p}{p}\right) = \left(\frac{n}{p}\right).$$

This shows:

Prop: Let $p \geqslant 3$ be an odd prime. Then

$$\chi_p(n) = \left(\frac{n}{p}\right)$$

defines a real character. Moreover, this is primitive.

Fact: Let $p \geqslant 3$ be an odd prime and χ is a real primitive character $(\text{mod } p^n)$.

Then $n=1$ and $\chi = \chi_p$.

Question: How about real primitive characters $(\text{mod } 2^n)$?

Ans/Fact: There are 3 real primitive character $(\text{mod } 2^n)$:

χ_{-4} $(\text{mod } 4)$ $\quad \chi_{-4}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \end{cases}$

χ_8 $(\text{mod } 8)$ $\quad \chi_8(n) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \pmod 8 \\ -1 & \text{if } n \equiv 3, 5 \pmod 8 \end{cases}$

χ_{-8} $(\text{mod } 8)$ $\quad \chi_{-8} = \chi_{-4}\chi_8$

$$= \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod 8 \\ -1 & \text{if } n \equiv 5, 7 \pmod 8 \end{cases}$$

Fact: Let $\chi \pmod q$ be a primitive character.

Suppose that $q = q_1 q_2$ and $(q_1, q_2) = 1$.

Then we can find primitive characters $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ such that $\chi = \chi_1 \chi_2$.

This implies:

Theorem: Every real primitive character is a product of.

$$\chi_4,\ \chi_8,\ \chi_p, \quad (p \geq 3, \text{ prime}).$$

Next, we want to find a "better" notation for real primitive characters.

Definition; Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be an odd positive integer and $n \in \mathbb{Z}$

We define the <u>Jacobi symbol</u>:

$$\left(\frac{n}{m}\right) = \left(\frac{n}{p_1}\right)^{\alpha_1} \cdots \left(\frac{n}{p_r}\right)^{\alpha_r}$$

Check: (1) $\left(\frac{ab}{m}\right) = \left(\frac{a}{m}\right)\left(\frac{b}{m}\right)$

(2) $\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right)$

(3) $\left(\frac{a+m}{m}\right) = \left(\frac{a}{m}\right)$

Theorem (Quadratic Reciprocity Law): Let m be an odd integer.

(1) $\left(\frac{-1}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4 \\ -1 & \text{if } m \equiv 3 \pmod 4 \end{cases}$

(2) $\left(\frac{2}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8 \\ -1 & \text{if } m \equiv 3, 5 \pmod 8 \end{cases}$

(3) Let m, n be odd positive integers.

$$\left(\frac{n}{m}\right)\left(\frac{m}{n}\right) = (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}}$$

Finally, we define the <u>Kronecker symbol</u>: for $u = \pm 1$

for $m = u \cdot 2^e \cdot p_1^{e_1} \cdots p_r^{e_r}$ and $n \in \mathbb{Z}$

$$\left(\frac{n}{m}\right) = \left(\frac{n}{u}\right) \cdot \left(\frac{n}{2}\right)^e \cdot \left(\frac{n}{p_1^{e_1} \cdots p_r^{e_r}}\right)$$

Here (1) $\left(\frac{n}{1}\right) = 1$

(2) $\left(\frac{n}{-1}\right) = \begin{cases} -1 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$

(3) $\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \equiv 1, 7 \pmod 8 \\ -1 & \text{if } n \equiv 3, 5 \pmod 8 \end{cases}$

We set $\left(\frac{n}{0}\right) = \begin{cases} 1 & \text{if } n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$

Warning: The property of the Kronecker symbol is <u>not</u> as good as that of previous symbols. Please be careful when you try to use it!

Definition: An integer D is a <u>fundamental discriminant</u> if (1) $D \equiv 1 \pmod 4$ and D is squarefree.
or (2) $D = 4d$, $d \equiv 2,3 \pmod 4$ and d is squarefree.

Theorem: Let χ be a real primitive character.
Then we can find a fundamental discriminant D such that:

$$\chi(n) = \left(\frac{D}{n}\right) \leftarrow \text{Kronecker symbol.}$$

Later, we will use the notation χ_D for it.

The conductor of χ_D is $|D|$.

Remark: When n is an odd positive integer.
The Kronecker symbol $\left(\frac{D}{n}\right)$ and the Jacobi symbol $\left(\frac{D}{n}\right)$ are the same.

Theorem': Let χ be a real primitive character mod $|D|$

Then $D = \chi(-1)\cdot|D|$ is a fundamental discriminant and for any odd integer n,

$$\chi(n) = \left(\frac{D}{n}\right) \leftarrow \text{Jacobi symbol.}$$