Let R be a commutative ring with identity 1.

Definition: A ring R is <u>Noetherian</u> if every ideal is finitely generated.

Proposition: TFAE

(1) A is Noetherian

(2) Every ascending chain of ideals:

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$$

eventually becomes constant, i.e, for some n, $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$

(3) Let S be a nonempty set of ideals. Then S contains a maximal element, i.e, we can find an ideal not properly contained in any other ideal in S

Proof: (1) $\Rightarrow$ (2) Assume that we have a chain:

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$$

Set $\mathfrak{a} = \bigcup_{i \geq 1} \mathfrak{a}_i$ This is an ideal.

A is Noetherian $\Rightarrow$ $\mathfrak{a}$ is finitely generated, by $x_1, \cdots x_m$

Then we can find $n \geq 1$ such that $x_1, \cdots x_m \in \mathfrak{a}_n$

This means: $\mathfrak{a} = \mathfrak{a}_n$ and hence $\mathfrak{a} = \mathfrak{a}_n = \mathfrak{a}_{n+1} = \mathfrak{a}_{n+2} = \cdots$

(2) ⇒ (3) Suppose not. Let $\mathfrak{a}_1 \in S$.

$\mathfrak{a}_1$ is not maximal $\Rightarrow$ can find $\mathfrak{a}_2$ s.t. $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2$

$\mathfrak{a}_2$ is not max'l $\Rightarrow$ can find $\mathfrak{a}_3$ s.t. $\mathfrak{a}_2 \subsetneq \mathfrak{a}_3$

Then we can find an ascending chain:

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \mathfrak{a}_4 \cdots$$

and this will never become constant. A contradiction!

(3) ⇒ (1) Let $\mathfrak{a} \subseteq A$ be an ideal.

$$S := \{ \text{finitely generated ideals} \subseteq \mathfrak{a} \}$$

S is nonempty, as zero ideal $\in S$.

Then S has a maximal element $\mathfrak{a}'$.

Claim: $\mathfrak{a}' = \mathfrak{a}$

Proof: Suppose not. Then $\mathfrak{a}' \subsetneq \mathfrak{a}$ and choose $x \in \mathfrak{a} - \mathfrak{a}$

Then $\langle \mathfrak{a}', x \rangle$ is finitely generated.

$\Rightarrow \langle \mathfrak{a}', x \rangle \in S$ and $\mathfrak{a}' \subsetneq \langle \mathfrak{a}', x \rangle$

A contradiction to that $\mathfrak{a}'$ is maximal.

$\mathfrak{a}' = \mathfrak{a}$, $\mathfrak{a}'$ finitely generated $\Rightarrow$ $\mathfrak{a}$ is finitely generated.

$\square$

Example: Euclidean domain $\Rightarrow$ PID $\Rightarrow$ Noetherian ring.

Definition: A (left **R-module** M is an abelian group with an operation: $R \times M \to M ; (r, m) \mapsto r.m$ satisfying:

(1) $(r_1 r_2).m = r_1.(r_2.m) \quad r_1, r_2 \in R,\ m \in M.$

(2) $1.m = m$

(3) $(r_1 + r_2).m = r_1.m + r_2.m, \quad r.(m_1 + m_2) = r.m_1 + r.m_2$

$r_1, r_2, r \in R, \quad m_1, m_2, m \in M.$

Remark: We can always view R as a R-module.

Definition: Let M be a R-module. $M' \subseteq M$ is a **R-submodule** if (1) M' is a subgroup of M

(2) for any $r \in R$ and $m' \in M'$, $r.m' \in M'$.

Remark: When consider R as R-module, the submodules are ideals.

Let $M' \subseteq M$ be a submodule. Then we can define the quotient module: let $m_1, m_2 \in M$

$m_1 \sim m_2$ iff $m_1 - m_2 \in M'$

This is an equivalent relation on M and

$M/M' := \{[m] : \text{equivalent classes under } "\sim"\}$

Check: This is a R-module via:

$$r.[m] = [r.m]$$

(This is well-defined sine $R \cdot M' \subseteq M'$)

Definition: Let M and N be two R-module. A <u>homomorphism</u> $\phi: M \to N$ is a group homomorphism satisfying:

$$\phi(r.m) = r.\phi(m) \quad \text{for } r \in R \text{ and } m \in M.$$

Remark: We can also formulate the module homomorphism theorems.

Example: Let $M' \subseteq M$ be a R-submodule. Then

$$\pi_{M'} : M \to M/M' \quad m \mapsto [m]$$

is a R-module homomorphism.

Let M be a R-module, and $S \subseteq M$ a subset.

We can construt a R-submodule generated by S:

$$\text{span}_R\langle S\rangle = \left\{ \sum r_i . s_i : \text{finite sum}, \ r_i \in R, \ s_i \in S \right\}$$

Check: This is a submodule of R.

Definition: A R-module M is finitely generated if we can find $S \subseteq M$ with $\# S < \infty$ and $M = \mathrm{span}_R \langle S \rangle$.

Question: Let M be a finitely generated R-module. Will its submodules finitely generated?

Ans: Generally this is not true. This is true for finitely generated R-module with R being Noetherian.

Definition: An R-module if Noetherian if every submodule is finitely generated.

Exercise: Let M be a R-module. TFAE

(1) M is Noetherian

(2) Every ascending chain of submodules eventually become constant

(3) Every nonempty set of submodules has a maximal element.

Proposition: Let M be a R-module. Suppose that $M' \subseteq M$ is a R-submodule such that both M' and M/M' are Neetherian. M is Noetherian.

Lemma: Let $M_1 \subseteq M_2$ be R-submodules of M.

Suppose that $M_1 \cap M' = M_2 \cap M'$ and $\pi_{M'}(M_1) = \pi_{M'}(M_2)$

Then $M_1 = M_2$

Proof: (Only need to show $M_1 \supseteq M_2$). Take $m_2 \in M_2$.

Then $\pi(M_1) = \pi(M_2) \Rightarrow$ can find $m_1 \in M$ such that

$$m_1 - m_2 \in M'$$

$M_1 \subseteq M_2 \Rightarrow m_1 - m_2 \in M_2 \Rightarrow m_1 - m_2 \in M' \cap M_2$

$M' \cap M_2 = M' \cap M_2 \Rightarrow m_1 - m_2 \in M_1 \Rightarrow m_2 \in M_1$ □

Proof of Prop: Let: $M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \subseteq \cdots$ be an ascending chain of submodules.

Then we have:

(1) $M_1 \cap M' \subseteq M_2 \cap M' \subseteq M_3 \cap M' \subseteq \cdots$

is ascending chain of submodules in M'

M' Neotherian, can find n s.t.

$$M_n \cap M' = M_{n+1} \cap M' = M_{n+2} \cap M' = \cdots$$

(2) $\pi_{M'}(M_1) \subseteq \pi_{M'}(M_2) \subseteq \pi_{M'}(M_3) \subseteq \cdots$

is an ascending chain of submodules in M/M'

M/M' Noetherian, can find m s.t.

$$\pi_{M'}(M_m) = \pi_{M'}(M_{m+1}) = \cdots$$

Next, take $l = \max(m, n)$ Then

$$M_l \cap M' = M_{l+1} \cap M' = M_{l+2} \cap M' = \cdots$$

$$\pi_{M'}(M_l) = \pi_{M'}(M_{l+1}) = \pi_{M'}(M_{l+2}) = \cdots$$

$$M_l \subseteq M_{l+1} \subseteq M_{l+2} \subseteq \cdots$$

By the lemma:

$$M_l = M_{l+1} = M_{l+2} = \cdots$$

This shows that the ascending chain will become constant. $\square$

Proposition: Let M be a finitely generated R-module with R being Noetherian. Then M is Noetherian.

Proof: Induction on the minimal # of generators of M.

Step I: Suppose that M is generated by one element, x

Then $M = \{r.x : r \in R\}$

We can construct a surjective module hommorphism

$$\begin{aligned} \phi: R &\to M \\ r &\mapsto r.x \end{aligned}$$

Then M is isomorphic to $R/\ker\phi$ as R-module.

Set $\tilde{\phi}: M \xrightarrow{\sim} R/\ker\phi$

Let $M' \subseteq M$ be a R-submodule. Then $\tilde{\phi}(M')$ is an ideal in $R/\ker(\phi)$

R Noetherian $\Rightarrow$ $R/\ker(\phi)$ is Noetherian

$\Rightarrow$ M' finitely generated $\Rightarrow$ M is Noetherian.

Step II. Let M be a finitely generated R-module with minimal # of generators n.

Then we find a submodule M' generated by $n-1$ elements and M/M' generated by 1 elements

Induction hypothesis $\Rightarrow$ M' Noetherian, M/M' Noetherian

$\Rightarrow$ M Noetherian. □