Let $d<0$ be a fundamental discriminant.

<u>Class number formula</u>:

$$L(1, \chi_d) = \frac{2\pi}{w\sqrt{|d|}} h(d) \qquad w = \begin{cases} 2 & d<-4 \\ 4 & d=-4 \\ 6 & d=-3 \end{cases}$$

Definition: Let $Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ be a quadratic form. An <u>automorphy</u> of Q is a $g \in SL_2(\mathbb{Z})$ s.t. $Q = {}^t g\, Q\, g$.

Proposition: Let $d<0$ be a fundamental discriminant. Then every quadratic form of discriminant d has exactly w automorphisms.

Observation: $g = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in SL_2(\mathbb{Z})} \qquad Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$

$$ {}^t g\, Q\, g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$= \begin{pmatrix} a\alpha^2 + b\alpha\gamma + c\gamma^2 & \frac{2a\alpha\beta + b\beta\gamma + b\alpha\delta + 2c\gamma\delta}{2} \\ \frac{2a\alpha\beta + b\beta\gamma + b\alpha\delta + 2c\gamma\delta}{2} & a\beta^2 + b\beta\delta + c\delta^2 \end{pmatrix}$$

Lemma: The set of automorphs of a given primitive quadratic form $Q = [a, b, c]$ (with $\mathrm{disc}(Q) = d$) equals:

$$\left\{ \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} : t, u \in \mathbb{Z},\ t^2 - du = 4 \right\} \subseteq SL_2(\mathbb{Z})$$

Note: $\frac{t-bu}{2} \cdot \frac{t+bu}{2} + acu^2 = \frac{t^2 - b^2u^2 + 4acu^2}{4} = \frac{t^2 - du}{4} = 1.$

Proof: We only show every automorph is of this form.

Set $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $\quad Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$

$${}^t g Q g = Q \Leftrightarrow \begin{cases} a\alpha^2 + b\alpha\gamma + c\gamma^2 = a \\ 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta = b \\ \alpha\beta^2 + b\beta\delta + c\delta^2 = c. \end{cases} \quad (*)$$

Notice that, $\mathrm{disc}(Q) = \mathrm{disc}({}^t g Q g)$, the last equality is redundant.

We also note: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad \alpha\delta + \beta\gamma = 1 + 2\beta\gamma$

$$(*) \Leftrightarrow \begin{cases} a\alpha^2 + b\alpha\gamma + c\gamma^2 = a \\ 2a\alpha\beta + b(1 + 2\beta\gamma) + 2c\gamma\delta = b \end{cases}$$

$$\Leftrightarrow \begin{cases} a\alpha^2 + b\alpha\gamma + c\gamma^2 = a & ① \\ a\alpha\beta + b\beta\gamma + c\gamma\delta = 0 & ② \end{cases}$$

$① \times \beta - ② \times \alpha.$

$$\rightsquigarrow \quad c\beta\gamma^2 - c\alpha\gamma\delta = a\beta \qquad \alpha\delta = \beta\gamma + 1.$$

$$\rightsquigarrow \quad a\beta + c\gamma = 0.$$

$① \times \delta - ② \times \gamma$

$$\rightsquigarrow \quad a\alpha^2\delta - a\alpha\beta\gamma + b\alpha\gamma\delta - b\beta\gamma^2 = a\delta$$

$$\rightsquigarrow \quad a\alpha(\alpha\delta - \beta\gamma) + b\gamma(\alpha\delta - b\beta) = a\delta$$

$$\rightsquigarrow \quad a\alpha + b\gamma = a\delta$$

$$\rightsquigarrow \quad a(\alpha - \delta) + b\gamma = 0$$

$$\Rightarrow \begin{cases} a\beta + c\gamma = 0 \\ a(\alpha - \delta) + b\gamma = 0 \end{cases} \Rightarrow a \mid c\gamma,\ b\gamma$$

Notice that Q is primitive $\Rightarrow (a, b, c) = 1 \Rightarrow a \mid \gamma.$

Then we write: $\gamma = au \Rightarrow \beta = -\frac{c\gamma}{a} = -cu.$

$$\alpha - \delta = -\frac{b\gamma}{a} = -bu$$

Then $1 = \alpha\delta - \beta\gamma = \alpha(\alpha + bu) + acu^2 = \alpha^2 + bu \cdot \alpha + acu^2$

$$\Rightarrow (2\alpha + bu)^2 + (4ac - b^2)u^2 = 4. \Downarrow$$

Then set $t = 2\alpha + bu \in \mathbb{Z}.$ $\qquad t^2 - du^2 = 4.$

Note: $\left.\begin{aligned} 2\alpha + bu &= t \\ \alpha - \delta &= -bu \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= \frac{t-bu}{2} \\ \delta &= \frac{t+bu}{2} \end{aligned}$

$$\Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \quad \text{with } \begin{cases} t^2 - du^2 = 4 \\ t, u \in \mathbb{Z}. \end{cases}$$

The inverse part is a direct calculation $\square$

Proof of Proposition:

When $d < -4$, $t^2 - du^2 = 4$ only has solutions $(\pm 2, 0)$

$\Rightarrow w = 2$

When $d = -4$, $t^2 - du^2 = 4 \rightsquigarrow t^2 + 4u^2 = 4$.

There are solutions: $(\pm 2, 0)$, $(0, \pm 1)$

$\Rightarrow w = 4$

When $d = -3$, $t^2 - du^2 = 4 \rightsquigarrow t^2 + 3u^2 = 4$.

Solutions: $(\pm 2, 0)$ $(\pm 1, \pm 1)$

$\Rightarrow w = 6$ $\square$.

Definition: Let Q be a quadratic form with $\operatorname{disc}(Q) = d$.

Let $n \geq 1$ be an integer.

$$R_Q(n) := \#\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : Q(x, y) = n \}$$

$R_Q^*(n) := \#\{(x,y) \in \mathbb{Z}\times\mathbb{Z} : Q(x,y) = n \text{ and } (x,y)=1\}$

If $Q(x,y)=n$, (x,y) is a representation of n.

$Q(x,y)=n$ with $(x,y)=1$, (x,y) is a proper representation of n.

Note: when $d<0$, $Q(x,y)$ is positive definite

$$\Rightarrow 0 \leqslant R_Q^*(n) \leqslant R_Q(n) < \infty.$$

Definition: Let d be a fundamental discriminant. Denote by S_d a set of quadratic forms s.t.

(1) $Q_1, Q_2 \in S_d \Rightarrow Q_1 \not\sim Q_2$.

(2) $\# S_d = h(d)$

Definition: Let $d \geqslant 1$ be a fundamental discriminat and $n \geqslant 1$ be an integer. Let S_d be a fixed set.

$$R(n;d) = \sum_{Q \in S_d} R_Q(n)$$

$$R^*(n;d) = \sum_{Q \in S_d} R_Q^*(n).$$

Note: $R(n;d)$, $R^*(n;d)$ are independent on the choice of S_d.

Theorem: Let $n>0$ and $(n,d)=1$. Then:

$$R(n;d) = w \sum_{m|n} \left(\frac{d}{m}\right)$$ ← Kronecker Symbol

Lemma: Assume $(n,d)=1$. There is a w-to-1 map from

$$M_1 = \{ \langle Q, x, y \rangle : Q \in S_d,\ Q(x,y)=n,\ (x,y)=1 \}$$

and

$$M_2 = \{ l : 0 \le l \le 2n-1,\ l^2 \equiv d \pmod{4n} \}$$

Proof: Take $\langle Q, x, y\rangle \in M_1$

$(x,y)=1 \Rightarrow$ can find $s,t \in \mathbb{Z}$ s.t. $xs-yt=1$.

Moreover, can find $s_0, t_0 \in \mathbb{Z}$ s.t. $\begin{cases} s = s_0 + hy \\ t = t_0 + hx \end{cases}$ $h \in \mathbb{Z}$.

$xs-yt=1 \Rightarrow \begin{pmatrix} x & t \\ y & s \end{pmatrix} \in SL_2(\mathbb{Z})$

Consider $Q \longrightarrow \begin{pmatrix} x & y \\ t & s \end{pmatrix} Q \begin{pmatrix} x & t \\ y & s \end{pmatrix} = [n, l, \frac{l^2-d}{4n}]$

Here $l = 2axr + b(xs+yr) + 2cys$.

$= 2axr_0 + b(xs_0+yr_0) + 2cys_0 + 2hn$.

Therefore, by modifying h, we can make $0 \le l \le 2n-1$

Then we get a map:

$F: M_1 \longrightarrow M_2 \quad \langle Q, x, y \rangle \mapsto l$.

Remark: If $F(\langle Q, x, y\rangle) = \ell$, we can find

$s, t \in \mathbb{Z}$ s.t.

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix} Q \begin{pmatrix} x & s \\ y & t \end{pmatrix} = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix} \qquad m = \frac{\ell^2 - d}{4n}.$$

① F is onto. Take $\ell \in M_2$, can find m s.t. $\ell^2 - d = 4nm$.

Then $\ell^2 - 4nm = d \Rightarrow [n, \ell, m]$ is a primitive $((n,d)=1)$ quadratic form of discriminant d.

Then we can find $Q \in S_d$ s.t. $Q \sim [n, \ell, m]$

That is, can find $g \in SL_2(\mathbb{Z})$ s.t. ${}^t g Q g = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix}$

Write $g = \begin{pmatrix} x & r \\ y & s \end{pmatrix} \Rightarrow Q(x, y) = n$.

$g \in SL_2(\mathbb{Z}) \Rightarrow (x, y) = 1$

$\Rightarrow \quad F(\langle Q, x, y\rangle) = \ell$.

② F is w-to-1.

Suppose that $F(\langle Q, x, y\rangle) = F(\langle Q', x', y'\rangle) = \ell$.

Then $Q \sim Q' \sim [n, \ell, \frac{\ell^2 - d}{4n}] \doteq [n, \ell, m]$

This will force $Q = Q'$, as S_d only contain inequivalent quadratic forms.

By Remark, for $\langle Q, x, y\rangle$, can find $\begin{pmatrix} x & r \\ y & s \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t.

$$\begin{pmatrix} x & y \\ r & s \end{pmatrix} Q \begin{pmatrix} x & r \\ y & s \end{pmatrix} = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix}$$

By Remark, for $\langle Q, x', y'\rangle$, can find $\begin{pmatrix} x' & r' \\ y' & s' \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t.

$$\begin{pmatrix} x' & y' \\ r' & s' \end{pmatrix} Q \begin{pmatrix} x' & r' \\ y' & s' \end{pmatrix} = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x & y \\ r & s \end{pmatrix} Q \begin{pmatrix} x & r \\ y & s \end{pmatrix} = \begin{pmatrix} x' & y' \\ r' & s' \end{pmatrix} Q \begin{pmatrix} x' & r' \\ y' & s' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x & r \\ y & s \end{pmatrix} \begin{pmatrix} x' & r' \\ y' & s' \end{pmatrix}^{-1} \text{ is an automophy of } Q$$

$\Rightarrow F$ is at most w-to-1.

Let $\langle Q, x, y\rangle \in M_1$, and $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is an automorphy,

Set $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix}$

Suppose that $F(\langle Q, x, y\rangle) = \ell$ Then

we can find r, s s.t. $\begin{pmatrix} x & y \\ r & s \end{pmatrix} Q \begin{pmatrix} x & r \\ y & s \end{pmatrix} = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix}$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an automophy $\Rightarrow$

$$\begin{pmatrix} x & y \\ r & s \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} Q \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x & r \\ y & s \end{pmatrix} = \begin{pmatrix} n & \frac{\ell}{2} \\ \frac{\ell}{2} & m \end{pmatrix}$$

$$(x \ y) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = (x', y') \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Note: $0 \leq l \leq 2n-1$

$\Rightarrow F(\langle Q, x', y' \rangle) = l = F(\langle Q, x, y \rangle)$

$\Rightarrow$ F is exactly n-to-1 $\square$