Recall: Let $d<0$ be a fundamental discriminant.

$$R(n;d) = \sum_{Q \in S_d} R_Q(n)$$

We showed: $\displaystyle R(n;d) = w \sum_{m|n} \left(\frac{d}{m}\right)$

when $n \geq 1$ and $(n,d)=1$.

Set $$G_d(N) = \frac{1}{wN} \sum_{\substack{1 \leq n \leq N \\ (n,d)=1}} R(n;d)$$

$$= \frac{1}{N} \sum_{\substack{1 \leq n \leq N \\ (n,d)=1}} \sum_{m|n} \left(\frac{d}{m}\right)$$

Proposition: $\displaystyle \lim_{N\to\infty} G_d(N) = \frac{\phi(|d|)}{|d|} \cdot L(1, \chi_d)$

Fact: $\displaystyle L(1,\chi_d) = \lim_{N\to\infty} \sum_{\substack{m=1 \\ (m,d)=1}}^{N} \frac{1}{m}\left(\frac{d}{m}\right)$

Lemma: As $N \to \infty$, $\displaystyle \sum_{\substack{1 \leq n \leq N \\ (n,d)=1}} 1 = \frac{\phi(|d|)}{|d|} \cdot N + O(|d|)$

Proof: Without loss of generality, we assume $N \geq |d|$.

Then we write $N = q|d| + r$ with $0 \leq r \leq |d|-1$

$$\underbrace{1, 2, \cdots |d|}_{|d|},\ \underbrace{d+1, \cdots 2|d|}_{|d|},\ \cdots\cdots,\ \underbrace{(q-1)|d|+1, \cdots q|d|}_{|d|},\ \underbrace{q|d|+1, \cdots q|d|+r}_{r}$$

Except the last set, each set has $\phi(|d|)$ numbes coprime to d.

Therefore, $\left| \sum_{\substack{1\le n\le N\\(n,d)=1}} 1 - q\phi(|d|) \right| \le r \le |d|$

$$q\phi(|d|) = q\cdot|d|\cdot\frac{\phi(|d|)}{|d|} = (N-r)\cdot\frac{\phi(|d|)}{|d|}$$

$$= \frac{\phi(|d|)}{|d|}\cdot N - r\frac{\phi(|d|)}{|d|}$$

and $r\cdot\frac{\phi(|d|)}{|d|} \le |d|$

$$\Rightarrow \left| \sum_{\substack{1\le n\le N\\(d,n)=1}} 1 - \frac{\phi(|d|)}{|d|} N \right| \le 2|d| \qquad \square.$$

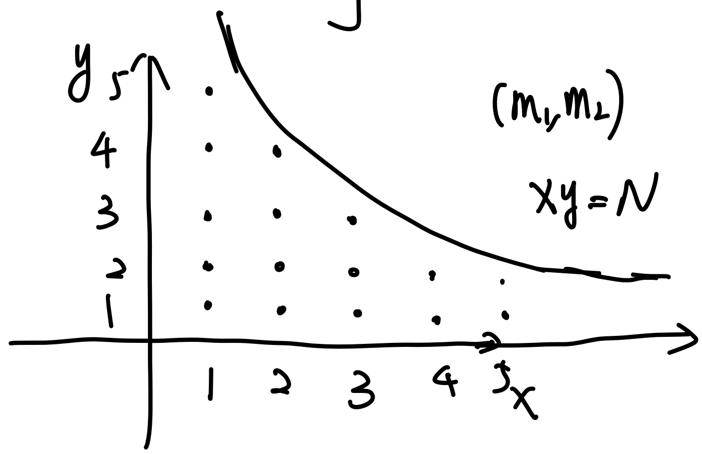
Proof of Proposition: We look at

$$(*) = \sum_{\substack{1\le n\le N\\(n,d)=1}} \sum_{m_1|n} \left(\frac{d}{m_1}\right) \qquad \left(G_d(N) = \frac{(*)}{N} \right)$$

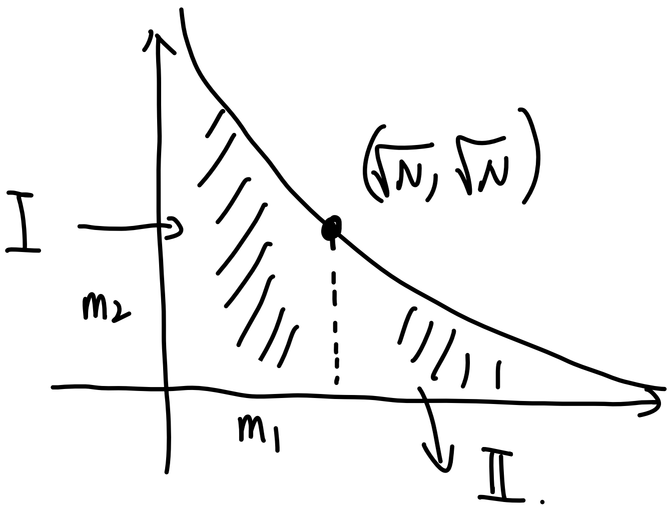
We write $n = m_1 m_2$. This becomes:

$$(*) = \sum_{\substack{1\le m_1 m_2\le N\\(m_1 m_2, d)=1}} \left(\frac{d}{m_1}\right)$$

This requires us to study $1 \leq m_1 m_2 \leq N$



The strategy:



I: $1 \leq m_1 \leq \sqrt{N}$, $1 \leq m_2 \leq \frac{N}{m_1}$

II: $1 \leq m_2 \leq \sqrt{N}$, $\sqrt{N} \leq m_1 \leq \frac{N}{m_2}$

$$\Rightarrow (*) = \sum_{\substack{1 \leq m_1 \leq \sqrt{N} \\ (m_1, d) = 1}} \left(\frac{d}{m_1}\right) \sum_{\substack{1 \leq m_2 \leq \frac{N}{m_1} \\ (m_2, d) = 1}} 1 \quad \leftarrow \text{I}$$

$$+ \sum_{\substack{1 \leq m_2 \leq \sqrt{N} \\ (m_2, d) = 1}} \sum_{\sqrt{N} \leq m_1 \leq \frac{N}{m_2}} \left(\frac{d}{m_1}\right) \quad \leftarrow \text{II}.$$

For I, apply the Lemma,

$$\sum_{\substack{1\le m_2\le \frac{N}{m_1}\\ (m_2,d)=1}} 1 = \frac{\phi(|d|)}{|d|}\cdot\frac{N}{m_1} + O(|d|)$$

$$\Rightarrow \mathrm{I} = \sum_{\substack{1\le m_1\le \sqrt{N}\\ (m_1,d)=1}} \left(\frac{d}{m_1}\right)\left(\frac{\phi(|d|)}{|d|}\,\frac{N}{m_1} + O(|d|)\right)$$

$$= \frac{\phi(|d|)\,N}{|d|}\cdot \sum_{\substack{1\le m_1\le \sqrt{N}\\ (m_1,d)=1}} \frac{\left(\frac{d}{m_1}\right)}{m_1} + O(\sqrt{N}\,|d|)$$

For II, $\left|\sum_{\sqrt{N}\le m_1\le \frac{N}{m_2}} \left(\frac{d}{m_1}\right)\right| \le |d|$

$$\Rightarrow |\mathrm{II}| \le \sum_{\substack{1\le m_2\le \sqrt{N}\\ (m_2,d)=1}} |d| \le |d|\cdot\sqrt{N} = O(\sqrt{N}\cdot|d|)$$

Therefore:

$$(*) = \mathrm{I} + \mathrm{II}$$

$$= \frac{\phi(|d)}{|d|}\cdot N\cdot \sum_{\substack{1\le m_1\le \sqrt{N}\\ (m_1,d)=1}} \frac{1}{m_1}\left(\frac{d}{m_1}\right) + O(\sqrt{N}\,|d|)$$

Then: $G_d(N) = \frac{\phi(|d|)}{|d|} \sum_{\substack{1 \le m_1 \le \sqrt{N} \\ (m_1, d)=1}} \frac{1}{m_1}\left(\frac{d}{m_1}\right) + O\left(\frac{|d|}{\sqrt{N}}\right)$

Then: $\lim_{N\to\infty} G_d(N) = \frac{\phi(|d|)}{|d|} \lim_{N\to\infty} \sum_{\substack{1 \le m_1 \le \sqrt{N} \\ (m_1, d)=1}} \frac{1}{m_1}\left(\frac{d}{m_1}\right) + 0$

$$= \frac{\phi(|d|)}{|d|} \cdot L(1, \chi_d) \qquad \square$$

Next, we count:

$$\sum_{\substack{1 \le n \le N \\ (n,d)=1}} R_Q(N) \qquad \text{with } Q=[a,b,c], \ (a,b,c)=1.$$

This is equal to the number of pairs $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ s.t.

(1) $1 \le ax^2+bxy+cy^2 \le N$

(2) $(ax^2+bxy+cy^2, d) = 1$.

We first consider this modulo d.

Lemma: $\#\{(x_0,y_0) : (x_0,y_0) \in (\mathbb{Z}/d\mathbb{Z})^2, \ ax_0^2+bx_0y_0+cy_0^2 \in (\mathbb{Z}/d\mathbb{Z})^\times\}$

$= |d| \cdot \phi(|d|)$

Proof: We write: $|d| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. By Chinese Remainder Theorem:

$$\#\{(x_0,y_0): (x_0,y_0)\in(\mathbb{Z}/d\mathbb{Z})^2,\ ax_0^2+bx_0y_0+cy_0^2\in(\mathbb{Z}/d\mathbb{Z})^\times\}$$

$$=\prod_{i=1}^{r}\#\{(x_0,y_0)\in(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^2 : p_i\nmid ax_0^2+bx_0y_0+cy_0^2\}$$

It suffices to show:

$$\#\{(x_0,y_0)\in(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^2 : p_i\nmid ax_0^2+bx_0y_0+cy_0^2\}=\phi(p_i^{\alpha_i})\,p_i^{\alpha_i}$$

Take p_i, then $p_i \mid d$. If $p_i\mid a$ and $p_i \mid c$

Then $d=b^2-4ac \Rightarrow p_i\mid b \Rightarrow p_i\mid a,b,c.$

This contradict to that $(a,b,c)=1 \Rightarrow \boxed{p_i\nmid a \text{ or } p_i\nmid c.}$

Case I: If p_i is odd and $p_i\nmid a$.

$$p_i\nmid(ax_0^2+bx_0y_0+cy_0^2)\iff p_i\nmid 4a(ax_0^2+bx_0y_0+cy_0^2)$$

$$\iff p_i\nmid(2ax_0+by_0)^2-dy_0^2$$

$$\iff p_i\nmid 2ax_0+by_0.$$

Set $2ax_0+by_0=t_0$ with $t_0\in(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$

Notice that $(2a,p_i)=1$. This equation has a unique solution for arbitrary $y_0\in\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$ and $t_0\in(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$

$$\Rightarrow \quad \#\left\{(x_0,y_0)\in\left(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}\right)^2 : p_i \nmid ax_0^2+bx_0y_0+cy_0^2\right\}$$

$$= p_i^{\alpha_i}\cdot \phi(p_i^{\alpha_i})$$

Case II: p_i odd, $p_i \nmid c$ multiply by $4c$ and a similar argument.

Case III: $p_i = 2$. $\Rightarrow 2 \mid d = b^2-4ac \Rightarrow 2 \mid b$.

$$2 \nmid ax_0^2+bx_0y_0+cy_0^2 \iff 2 \nmid ax_0^2+cy_0^2$$

$$\iff 2 \mid (ax_0+cy_0)^2$$

$$\iff 2 \nmid ax_0+cy_0. \quad (**)$$

$2 \mid b \Rightarrow 2\nmid a$ or $2 \nmid c$ $\quad ((a,b,c)=1)$

$2 \nmid a$ choose x_0 to be odd and y_0 arbitrary

This always satisfy $(**)$

The # of $(x_0,y_0) = 2^{\alpha_i}(2-1)\cdot 2^{\alpha_i} = \phi(2^{\alpha_i})\cdot 2^{\alpha_i}$

$2 \nmid c$ choose y_0 to be odd and x_0 even.

The # of $(x_0,y_0) = 2^{\alpha_i}(2-1)\cdot 2^{\alpha_i} = \phi(2^{\alpha_i})\cdot 2^{\alpha_i}$ $\square$

Proposition: $\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le n\le N \\ (n,d)=1}} R_Q(n) = \frac{\phi(|d|)}{|d|} \frac{2\pi}{|d|^{\frac{1}{2}}}$

This imply:

$$\lim_{N\to\infty} G_d(N) = \lim_{N\to\infty} \frac{1}{wN} \sum_{Q\in S_d} \sum_{\substack{1\le n\le N \\ (n,d)=1}} R_Q(n)$$

$$= \frac{1}{w} \sum_{Q\in S_d} \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le n\le N \\ (n,d)=1}} R_Q(n)$$

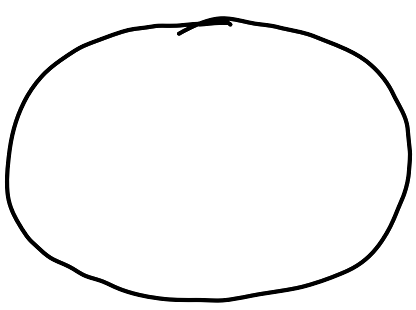
$$= \frac{1}{w} \sum_{Q\in S_d} \frac{\phi(|d|)}{|d|} \frac{2\pi}{|d|^{\frac{1}{2}}}$$

$$= \frac{\phi(|d|)}{|d|} \cdot \frac{2\pi}{w|d|^{\frac{1}{2}}} h(d)$$

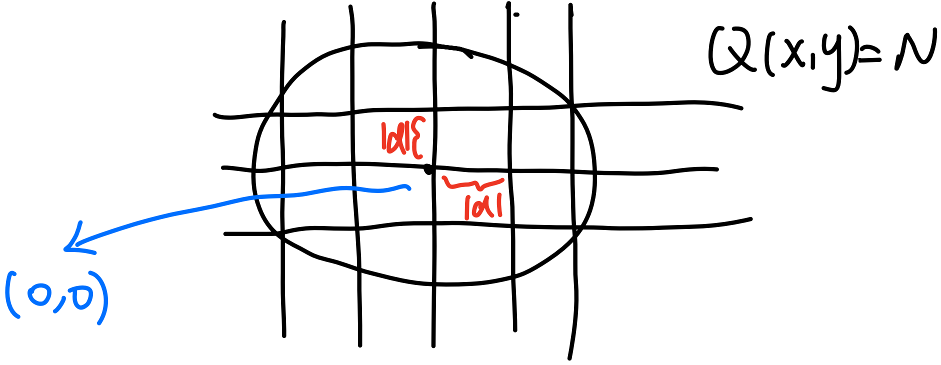
Proof of Proposition: Let $Q = ax^2 + bxy + cy^2$

This is an ellipse. when $d = b^2 - 4ac < 0$ and $a > 0$.

We draw: $Q(x,y) = N$ $\quad ax^2 + bxy + cy^2 = N$.



Then we slice the ellipse into small pieces:



such that each small piece is a square with area d^2.

Then:

$$\sum_{\substack{1\le n\le N\\(n,d)=1}} R_Q(n) = \#\bigsqcup_{\text{squares}} \{(x_0,y_0)\in \text{box},\ d\nmid Q(x_0,y_0),\ Q(x_0,y_0)\le N\}$$

$$= \#\bigsqcup_{\text{squares}} \{(x_0,y_0)\in (\mathbb{Z}/d\mathbb{Z})^2 : d\nmid Q(x_0,y_0)\}$$

$$= \#\bigsqcup_{\text{squares}} |d|\cdot\phi(|d|) = |d|\cdot\phi(|d|)\cdot \#\text{squares}$$

$$\#\text{squares} \approx \frac{\text{Area of } Q(x,y)=N}{d^2}$$

$$\lim_{N\to\infty}\frac{1}{N}\#\text{squares} = \lim_{N\to\infty}\frac{\text{Area of } Q(x,y)=N}{N\,|d|^2}$$

$$\text{Area}(Q(x,y)=N) = \frac{2\pi}{|d|^{\frac{1}{2}}}$$

$$\Rightarrow \lim_{N\to\infty}\sum_{1\le n\le N} R_Q(n) = \frac{2\pi}{|d|^{\frac12}}\cdot\frac{1}{|d|^2}\cdot\phi(|d|)\cdot|d|$$

$$= \frac{\phi(|d|)}{|d|}\cdot\frac{2\pi}{|d|^{\frac12}} \qquad \square$$

Note: $Q(x,y) = N = ax^2 + bxy + cy^2$

Fact: An ellipse $Ax^2 + Bxy + Cy^2 = 1$

has area: $\dfrac{2\pi}{\sqrt{4AC - B^2}}$

$$\text{area} = \frac{2\pi}{\sqrt{\dfrac{4ac-b^2}{N^2}}} = \frac{2\pi N}{|d|^{\frac{1}{2}}}$$