(Chapter 26 P 199-202)

Many number theoretic assertions have the form:

$\boxed{\text{such and such a statement is true for every natural number } n.}$

Here are some examples:

- $1^2+2^2+3^2+\cdots+n^2 = \dfrac{2n^3+3n^2+n}{6}$ for every $n \in \mathbb{N}$.

- Every natural number n is equal to a sum of at most four squares.

If we want to check these statements for any particular integer, this is easy.

For example:

✓ $1^2+2^2+3^2+4^2 = 1+4+9+16 = 30$

$$\frac{2(4)^3+3(4)^2+4}{6} = \frac{128+48+4}{6} = \frac{180}{6} = 30$$

✓ $4 = 1^2+1^2+1^2+1^2$

However, this is <u>not</u> a proof since we want to show the statement is true for all natural numbers.

<u>Remark</u>: In a lot of cases,

to prove something is hard, since you need to show it for all the cases.

to disprove something is easy, since you only need to show it is false for one example.

Therefore, we need to study the method of proof.

Here are 3 main methods we will use:

(1) proof by using definition.

(2) proof by contradiction.

(3) induction.

In today's class, we will study the <u>induction method</u>.

Induction method:

Step I: (Initialization) Check the initial case $n=1$.

Step II: (Induction step) Assume that we have already completed the ~~proof~~ for n. (called induction hypothesis), prove the statement for $n+1$.

Why this methods work?

Reason: By step I, $n=1$ ✓

by step II, $n=2$ ✓

by step II, $n=3$ ✓

$\vdots$

Domino effect

Example: Use induction to show:

$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n + n}{6} \quad \text{for all } n.$$

Proof: We write: for all n

$$P(n): \quad 1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n + n}{6}.$$

step I: show $P(1)$ is true

step II: assume that $P(n)$ is true

show $P(n+1)$ is true

Step I: $P(1)$ $\quad 1^2 = 1$

$$\frac{2(1)^3 + 3(1) + 1}{6} = \frac{6}{6} = 1 \qquad \checkmark$$

Step II: Assume $P(n)$ is true, that is

$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

$$\left(P(n+1): \ 1^2 + \cdots n^2 + (n+1)^2 = \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6}\right)$$

$$\underbrace{1^2 + \cdots n^2}_{} + (n+1)^2$$

$$\underline{\underline{\text{by induction}}} \ \frac{2n^3+3n^2+n}{6} + (n+1)^2$$

$$= \frac{2n^3+3n^2+n}{6} + n^2+2n+1$$

$$= \frac{2n^3+3n^2+n+6n^2+12n+6}{6}$$

$$= \frac{2n^3+9n^2+13n+6}{6}$$

$$\frac{2(n+1)^3+3(n+1)^2+(n+1)}{6} = \frac{2(n^3+3n^2+3n+1)+3(n^2+2n+1)+(n+1)}{6}$$

$$= \frac{2n^3+9n^2+13n+6}{6} \quad \checkmark$$

Therefore, by induction, we show $P(n)$ is true for all n, that is,

for all n, $1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}$ $\square$

Another version of induction:

<u>complete induction/strong induction:</u>

step I: show that $n=1$ is true

step II: assume that the statement is true for numbers up to n, show that $n+1$ is true.