(Chapter 38 P 313 – 315)

We look at the powers of $A+B$

$$(A+B)^0 = 1$$
$$(A+B)^1 = A+B$$
$$(A+B)^2 = A^2+2AB+B^2$$
$$(A+B)^3 = A^3+3A^2B+3AB^2+B^3$$
$$(A+B)^4 = A^4+4A^3B+6A^2B^2+4AB^3+B^4$$

\ - - -

We take the coefficients for RHS and we got

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

In this chapter, we want to investigate what happens when $(A+B)^n$ is multiplied out.

It should be of the form

$$(A+B)^n = \square A^n + \square A^{n-1}B + \square A^{n-2}B^2 + \cdots \square A^{n-k}B^k + \cdots \square A^2B^{n-2} + \square AB^{n-1} + \square B^n$$

<u>Definition</u>: The integers showing up in the expansion of $(A+B)^n$ are called <u>binomial coefficients</u>.

More precisely, let n be a natural number and k is another integer satisfying $0 \le k \le n$.

Then the binomial coefficient

$$\binom{n}{k} = \text{coefficient of } A^{n-k}B^k \text{ in } (A+B)^n$$

$$(A+B)^n = \square A^n + \square A^{n-1}B + \square A^{n-2}B^2 + \cdots \square A^{n-k}B^k + \cdots \square A^2B^{n-2} + \square AB^{n-1} + \square B^n$$

Therefore, we can write:

$$(A+B)^n = \binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \cdots \binom{n}{k}A^{n-k}B^k$$
$$+ \cdots \binom{n}{n-1}AB^{n-1} + \binom{n}{0}B^n.$$

The Pascal's triangle is:

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ \cdots & & & & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ \cdots & & & & & & \end{array}$$

A <u>natural</u> question: how to calculate $\binom{n}{k}$?

The calculation can be based on the following facts:

Fact 1: $\binom{n}{0} = \binom{n}{n} = 1.$

Fact 2: (Theorem 38.1: addition formula for binomial coefficients)

Suppose that $0 \le k \le n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Fact 3: $\binom{n}{k} = \binom{n}{n-k}$

Proof: (1) We prove this by induction:

$$P(n): \quad \binom{n}{0} = \binom{n}{n} = 1$$

Step I: Check $P(1)$. This is true sine

$$A+B = 1 \cdot A + 1 \cdot B$$

Step II: Assume $P(n)$.

$$(A+B)^{n+1} = (A+B)(A+B)^n$$

$$= A(A+B)^n + B(A+B)^n$$

$$= A\left(\binom{n}{0} A^n + \binom{n}{1} A^{n-1}B + \cdots + \binom{n}{n-1} AB^{n-1} + \binom{n}{n} B^n \right)$$

$$+ B\left(\binom{n}{0} A^n + \binom{n}{1} A^{n-1}B + \cdots + \binom{n}{n-1} AB^{n-1} + \binom{n}{n} B^n \right)$$

$$= \binom{n}{0} A^{n+1} + \binom{n}{1} A^n B + \cdots \binom{n}{n} AB^n$$

$$+ \binom{n}{0} A^n B + \cdots \binom{n}{n-1} AB^n + \binom{n}{n} B^{n+1}$$

$$= \binom{n}{0} A^{n+1} + \_\_\_ A^n B + \cdots + \_ AB^n + \binom{n}{n} B^{n+1}$$

Recall: $\binom{n+1}{0}$ = coefficient of A^{n+1} $\Rightarrow \binom{n+1}{0} = \binom{n}{0} = 1$

$\binom{n+1}{n+1}$ = coefficient of B^{n+1} $\qquad \binom{n+1}{n+1} = \binom{n}{n} = 1$

This shows: $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$ ($P(n+1)$ is true.)

Therefore, by math induction, $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$. $\square$

Proof of Fact 2:

$$
\begin{aligned}
(A+B)^{n+1} &= (A+B)(A+B)^n = A(A+B)^n + B(A+B)^n \\
&= A\left(\binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \cdots + \binom{n}{k}A^{n-k}B^k + \cdots \binom{n}{n}B^n\right) \\
&\quad + B\left(\binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \cdots + \binom{n}{k-1}A^{n-k+1}B^{k-1} + \cdots \binom{n}{n}B^n\right) \\
&= \binom{n}{0}A^{n+1} + \binom{n}{1}A^nB + \cdots \binom{n}{k}A^{n-k+1}B^k + \cdots \binom{n}{n}AB^n \\
&\quad + \binom{n}{0}A^nB + \cdots \binom{n}{k-1}A^{n-k+1}B^k + \cdots \binom{n}{n-1}AB^n + \binom{n}{n}B^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
\binom{n+1}{k} &= \text{the coefficient of } A^{n+1-k}B^k \\
&= \binom{n}{k} + \binom{n}{k-1}
\end{aligned}
$$

$\square$

Proof of Fact 3: By definition:

$$(A+B)^n = \binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \cdots \underline{\binom{n}{k}A^{n-k}B^k} + \cdots \binom{n}{n-1}AB^{n-1} + \binom{n}{n}B^n.$$

We change the order of A, B

$$(B+A)^n = \binom{n}{0}B^n + \binom{n}{1}B^{n-1}A + \cdots \underline{\binom{n}{n-k}B^kA^{n-k}} + \cdots \binom{n}{n-1}BA^{n-1} + \binom{n}{n}A^n$$

Since $(A+B)^n = (B+A)^n$ and we can choose any number for A, B,

$$\binom{n}{k}A^{n-k}B^k = \binom{n}{n-k}B^kA^{n-k}.$$

This implies: $\binom{n}{k} = \binom{n}{n-k}$ $\square$.

<u>Remark</u>: By Fact 1 and Fact 2, we can get Pascal's triangle:

Step 1:
(Fact 1)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & & & 1 & \\ \cdot^{\cdot^{\cdot}} & & & & & & \ddots \end{array}$$

Step II

(Part 2.

1

1 1

1 2. 1

1 3 3 1

1 4 6 4. 1

- - - -