(Chapter 13.)

We proved, there are <u>infinitely</u> many primes.

We want to describe the number of primes in a more precise way.

Therefore, we will introduce the counting function.

Before that, we look at one easier counting function.

For $x > 0$, we define:

$$E(x) := \#\{\text{even numbers } n \text{ with } 1 \leq n \leq x\}$$

Example:

$$E(1) = \#\{\text{even numbers } n \text{ with } 1 \leq n \leq 1\} = 0.$$

$$E(3) = \#\{\text{even numbers } n \text{ with } 1 \leq n \leq 3\}$$
$$= \#\{2\} = 1$$

$$E(4) = \#\{\text{even numbers } n \text{ with } 1 \leq n \leq 4\}$$
$$= \#\{2, 4\} = 2$$

...

$$E(51) = \#\{\text{even numbers } n \text{ with } 1 \leq n \leq 51\}$$
$$= \#\{2, 4, 6, 8, \cdots 48, 50\} = 25$$

$E(100) = \#\{\text{even numbers } n \text{ with } 1 \le n \le 100\}$

$= \#\{2, 4, 6, \cdots 98, 100\} = 50.$

We look at $\frac{E(x)}{x}$.

$\frac{E(1)}{1} = 0$, $\frac{E(3)}{3} = \frac{1}{3}$, $\frac{E(4)}{4} = \frac{1}{2}$,

$\frac{E(51)}{51} = \frac{25}{51}$, $\frac{E(100)}{100} = \frac{50}{100}.$

Indeed, we can show, as x becomes larger and larger, $\frac{E(x)}{x}$ is closer and closer to $\frac{1}{2}$.

Therefore, we have:

$$\lim_{x \to \infty} \frac{E(x)}{x} = \frac{1}{2}. \quad \left(\text{or } E(x) \sim \frac{x}{2}.\right)$$

Next, we consider prime counting function:

$\pi(x) := \#\{\text{primes } p \text{ with } 1 \le p \le x\}$

Example $\pi(10) = \#\{\text{primes } p \text{ with } 1 \le p \le 10\}$

$= \#\{2, 3, 5, 7\} = 4.$

Theorem 13.1 (Prime Number Theorem, PNT)

$$\lim_{x \to} \frac{\pi(x)}{\frac{x}{\ln x}} = 1 \quad \left(\pi(x) \sim \frac{x}{\ln x} \right)$$

- This was conjectured by Gauss and Legendre. in 1800s.
- This was proved by Hadamard and de la Vallée Poussin independently.
- In their proofs, they used methods in complex analysis (calculus for complex numbers) to study Riemann zeta funtion.

  It's suprising that, we need calculus to study integers.

  This is one of the branches of number theory — analytic number theory.

- In 1948, Erdös and Selberg found an elementary proof for PNT