Denote by $i = \sqrt{-1}$. The <u>complex numbers</u> are defined by

$$\mathbb{C} = \{ z = a + bi \mid a, b \in \mathbb{R} \}$$

Let $a+bi,\ c+di \in \mathbb{C}$, we define:

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi)\cdot(c+di) = (ac-bd) + (ad+bc)i.$$

Theorem: $(\mathbb{C}, +, \cdot)$ is a field.

Before the proof, we introduce the <u>conjugate</u> of a complex number: let $z = a+bi$ with $a, b \in \mathbb{R}$, then the <u>conjugate</u> of z, denoted by $\bar{z}$, is $\bar{z} = a - bi$

Observation:
$$z \cdot \bar{z} = (a+bi)\cdot(a-bi)$$
$$= (a^2+b^2) + (a(-b)+ab)i = a^2+b^2 \in \mathbb{R}. \quad \geq 0.$$

The <u>norm</u> of z, denoted by $|z|$, is:

$$|z| = \sqrt{a^2+b^2} = (z\cdot\bar{z})^{\frac{1}{2}}.$$

If $z \neq 0$, $a, b \neq 0$, then $|z| > 0$.

This also shows: , for $z \neq 0$, $z \cdot \frac{\bar{z}}{|z|^2} = \frac{z \cdot \bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$.

Proof of Theorem:

(1) For $+$:

(a) (Associative) let $x_1 + y_1 i$, $x_2 + y_2 i$, $x_3 + y_3 i \in \mathbb{C}$,

$$x_1 + y_1 i + \left((x_2 + y_2 i) + (x_3 + y_3 i) \right)$$
$$= x_1 + y_1 i + (x_2 + x_3) + (y_2 + y_3) i$$
$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) i$$

$$\left((x_1 + y_1 i) + (x_2 + y_2 i) \right) + (x_3 + y_3 i)$$
$$= (x_1 + x_2) + (y_1 + y_2) i + x_3 + y_3 i$$
$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) i. \quad \checkmark$$

(b) (Commutative) $x + yi$, $a + bi \in \mathbb{C}$

$$(x + yi) + (a + bi) = (x + a) + (y + b) i$$
$$= (a + x) + (b + y) i$$
$$= (a + bi) + (x + yi) \quad \checkmark$$

(c) (Additive identity) : we can show:

$$a + bi + 0 = (a + 0) + bi = a + bi \quad \checkmark$$

(d) (Additive inverse) : for $a+bi \in \mathbb{C}$,

$$a+bi + (-a-bi) = (a-a) + (b-b)i = 0 + 0i = 0$$

(2) For $\cdot$

(a) (Associative)

$$(a+bi)\big((c+di)(x+yi)\big)$$

$$= (a+bi)\big(cx-dy + (cy+dx)i\big)$$

$$= a(cx-dy) - b(cy+dx)$$

$$+ b(cx-dy)i + a(cy+dx)i$$

$$= (acx - ady - bcy - bdx)$$

$$+ (bcx - bdy + acy + adx)i$$

$$\big((a+bi)(c+di)\big)(x+yi)$$

$$= \big((ac-bd) + (ad+bc)i\big)(x+yi)$$

$$= (ac-bd)x - y(ad+bc)$$

$$+ (ac-bd)yi + (ad+bc)xi$$

$$= (acx - bdx - yad - ybc)$$

$$+ (acy - bdy + adx + bcx)i \quad \checkmark$$

(b) Commutative: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$(c+di)(a+bi) = ca - db + (da+cb)i$

$= ac - bd + (ad+bc)i$ ✓

(c) (Multiplicative identity) $(a+bi)\cdot 1 = (a+bi)(1+0\cdot i)$

$= (a - b\cdot 0) + (a\cdot 0 + b\cdot 1)i$

$= a+bi$ ✓

(d) (Multiplicative inverse) $a+bi \neq 0 \Rightarrow a,b \neq 0$

$(a+bi)\cdot\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right)$

$= \left(a\cdot\frac{a}{a^2+b^2} - b\left(-\frac{b}{a^2+b^2}\right)\right) + \left(a\cdot\left(\frac{-b}{a^2+b^2}\right) + b\left(\frac{a}{a^2+b^2}\right)\right)i$

$= \frac{a^2+b^2}{a^2+b^2} = 1$

(3) $(a+bi)(x_1+y_1 i + x_2+y_2 i)$

$= (a+bi)((x_1+x_2) + (y_1+y_2)i)$

$= a(x_1+x_2) - b(y_1+y_2) + (a(y_1+y_2) + b(x_1+x_2))i$

$(a+bi)(x_1+y_1 i) + (a+bi)(x_2+y_2 i)$

$= (ax_1 - by_1) + (ay_1 + bx_1)i + (ax_2 - by_2) + (ay_2 + bx_2)i$

$$= (ax_1 + ax_2 - by_1 - by_2) + (ay_1 + ay_2 + bx_1 + bx_2)i \quad \checkmark.$$