The Pigeonhole Principle.

Suppose that there are $n+1$ pigeons but n pigeonholes then at least one pigeonhole contains more than one pigeon.

Theorem 33.1 Dirichlet's Diophantine Approximation Theorem.

Let D be a positive integer that is not a square. Then there are infinitely many pairs of positive integers (x, y) such that

$$\left| x - y\sqrt{D} \right| < \frac{1}{y}.$$

Remark: (1) When D is not a square, $\sqrt{D}$ is irrational. This is the only place where we use that D is not a square. Indeed, this theorem can be generalized to any <u>irrational</u> numbers

(2) Here is another way to write

$$\left| x - y\sqrt{D} \right| < \frac{1}{y}$$

that is:

$$\left| \frac{x}{y} - \sqrt{D} \right| < \frac{1}{y}.$$

This means: for $\sqrt{D}$ (irrational), we can always find rational numbers $\frac{x}{y}$ that is close to $\sqrt{D}$. That's why we say this is an "approximation."

Proof of Theorem: Let Y be a large integer.

We investigate:

$Y+1$ numbers.

$$\begin{cases} 0\cdot\sqrt{D} = N_0 + F_0 & N_0=0,\ F_0 \\ 1\cdot\sqrt{D} = N_1 + F_1 & N_1 \text{ integer},\ 0\le F_1<1 \\ & (N_1=\lfloor\sqrt{D}\rfloor) \\ 2\cdot\sqrt{D} = N_2 + F_2, & N_2 \text{ integer},\ 0\le F_2<1 \\ & (N_2=\lfloor 2\sqrt{D}\rfloor) \\ \quad\cdots & \\ Y\cdot\sqrt{D} = N_Y + F_Y & N_Y \text{ integer},\ 0\le F_Y<1 \\ & (N_Y=\lfloor Y\sqrt{D}\rfloor) \end{cases}$$

$\Rightarrow$ $F_0, F_1, \ldots F_Y$ are $Y+1$ integers (pigeons) in $[0, 1)$.

We divide $[0, 1)$ into several smaller intervals (pigeoholes)

$[0, \frac{1}{Y})$, $[\frac{1}{Y}, \frac{2}{Y})$, $[\frac{2}{Y}, \frac{3}{Y})$, $\cdots$ $[\frac{Y-1}{Y}, 1)$

$F_0, F_1, \cdots F_Y$ ($Y+1$ pigeons) must fall into.

Y interals (Y pigeonholes)

By Pigeonhole principal, we can find $0 \le m < n \le Y$

such that F_m and F_n is the same interval

(same pigeon hole)

This shows: $|F_m - F_n| < \frac{1}{Y}$

since each interval is of length $\frac{1}{Y}$.

$$m\sqrt{D} = N_m + F_m \Rightarrow F_m = m\sqrt{D} - N_m$$

$$n\sqrt{D} = N_n + F_n \Rightarrow F_n = n\sqrt{D} - N_n$$

and $N_m < N_n$

$$\Rightarrow |F_m - F_n| = |(N_n - N_m) - (n-m)\sqrt{D}| < \frac{1}{Y}.$$

Then set $x = N_n - N_m$ and $y = n - m$

Since $0 \le m < n \le Y \Rightarrow y = n - m \le Y \Rightarrow y \le Y \Rightarrow \frac{1}{Y} < \frac{1}{y}$.

This becomes $|x - y\sqrt{D}| < \frac{1}{y}$.

We next show: there are infinitely many pairs. Suppose not, we can find $(x_1, y_1) \cdots (x_n, y_n)$ with

$$|x_i - y_i \sqrt{D}| < \frac{1}{y_i} \qquad i=1,2,\cdots n$$

Then we can find a such that:

$$|x_1 - y_1\sqrt{D}|, |x_2 - y_2\sqrt{D}|, \cdots |x_n - y_n\sqrt{D}| > a.$$

Then we can find an integer Y' such that

$$\frac{1}{Y'} > a$$

We run the argument above before, and we can find (x_0, y_0)

$$|x_0 - y_0\sqrt{D}| < \frac{1}{y_0} < \frac{1}{Y'} < a$$

The last "$<$" guarantees that (x_0, y_0) is a new pair.

A contradiction! $\square$

Next, we replace $\sqrt{D}$ by any irrational integers:

Theorem 33.2 (Dirichlet's Diophantine Approximation Theorem)

Let $\alpha > 0$ be an irrational number. Then there are infinitely many pairs of positive integers (x, y) such that

$$|x - y\alpha| < \frac{1}{y}.$$