Recall: Let $\alpha \in \mathbb{C}$.

α is algebraic if we can find $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$

Otherwise, α is transcendental.

In this lecture, we will prove:

$$\beta = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.11001\ldots \quad \text{Liouville's number.}$$

is transcendental.

Theorem (Liouville's Inequality) Let α be an algebraic number and α is a root of the polynomial

$$f(x) = C_0 x^d + C_1 x^{d-1} + \cdots C_{d-1} x + C_d$$

with integral coefficients. Let D be any number with $D > d$.

Then there are only finitely many rationals $\frac{a}{b}$ $(a, b \in \mathbb{Z})$ that satisfy the inequality:

$$\left| \frac{a}{b} - \alpha \right| \leq \frac{1}{b^D}.$$

Remark: Compare this with Dirichlet's Diophantine Approximation Theorem!

We use an example to highlight the idea of proof:

Consider $x = \sqrt{2}$ $\quad f(x) = x^2 - 2$.

Suppose that $D > 2$ and $\frac{a}{b}$ is a rational satisfying

$$\left| \frac{a}{b} - \sqrt{2} \right| < \frac{1}{b^D}.$$

Idea: find an upper bound for $\left| f\left(\frac{a}{b}\right) \right|$

a lower bound for $\left| f\left(\frac{a}{b}\right) \right|$.

Upper bound: $f(x) = x^2 - 2 = (x - \sqrt{2})(x + 2)$

$$\left| f\left(\frac{a}{b}\right) \right| = \left| \frac{a}{b} - \sqrt{2} \right| \cdot \left| \frac{a}{b} + \sqrt{2} \right|$$

$$\left| \frac{a}{b} - \sqrt{2} \right| < \frac{1}{b^D} < 1 \Rightarrow \left| \frac{a}{b} \right| < \sqrt{2} + 1 < 3.$$

$$\Rightarrow \left| f\left(\frac{a}{b}\right) \right| = \left| \frac{a}{b} - \sqrt{2} \right| \left| \frac{a}{b} + \sqrt{2} \right|$$

$$< \frac{1}{b^D} \cdot 3 = \frac{3}{b^D}.$$

Lower bound: $f(\frac{a}{b}) = (\frac{a}{b})^2 - 2 = \frac{a^2}{b^2} - 2 = \frac{a^2-2b^2}{b^2}$.

Notice: $f(\frac{a}{b}) \neq 0. \Rightarrow a^2-2b^2 \neq 0. \Rightarrow |a^2-2b^2| \geq 1.$

$$\Rightarrow \left|f(\frac{a}{b})\right| = \left|\frac{a^2-2b^2}{b^2}\right| \geq \frac{1}{b^2}.$$

Combine two results:

$$\frac{1}{b^2} \leq \left|f(\frac{a}{b})\right| \leq \frac{3}{b^D}$$

$$\Rightarrow \quad b^{D-2} \leq 3$$

$D>2 \Rightarrow \quad b = 1, 2. \quad \rightsquigarrow$ finitely many b.

For each b, there are at most finitely many choice of a since $|\frac{a}{b} - \sqrt{2}| < 1$.

Therefore, there are only finitely many choice of $\frac{a}{b}$ such that $|\frac{a}{b} - \sqrt{2}| < \frac{1}{b^D}$. $\square$

Proof of Liouville's Inequality:

Lemma: Let α be a solution for a polynomial $f(x) \in \mathbb{Z}[x]$, then

$$f(x) = (x-\alpha)\cdot g(x)$$

We skip the proof of this lemma.

We write: $g(x) = e_1 x^{d-1} + e_2 x^{d-2} + \dots e_{d-1} x + e_d$.

Note: $e_1, \dots e_d$ are <u>not</u> necessarily integers.

Now: assume $D > d = \deg f(x)$.

and $\frac{a}{b}$ satisfies: $\left|\frac{a}{b} - \alpha\right| < \frac{1}{b^D}$. (<1)

(and hence $\left|\frac{a}{b}\right| \leq |\alpha| + 1$)

<u>Upper bound</u>: $f(x) = (x-\alpha)\cdot g(x)$

$$\left|f\left(\frac{a}{b}\right)\right| = \left|\frac{a}{b} - \alpha\right| \cdot \left|g\left(\frac{a}{b}\right)\right|$$

Triangle's inequality: $y, z \in \mathbb{R}$, $|y+z| \leq |y| + |z|$

$$\left|g\left(\frac{a}{b}\right)\right| = \left|e_1\left(\frac{a}{b}\right)^{d-1} + \dots e_{d-1}\left(\frac{a}{b}\right) + e_d\right|$$

$$\leq |e_1|\left|\frac{a}{b}\right|^{d-1} + |e_2|\left|\frac{a}{b}\right|^{d-2} + \dots |e_{d-1}|\left|\frac{a}{b}\right| + |e_d|$$

$$\leq |e_1|\left(|\alpha|+1\right)^{d-1} + |e_2|\left(|\alpha|+1\right)^{d-2} + \dots |e_{d-1}|\left(|\alpha|+1\right) + |e_d|$$

$\leq K.$ a constant.

This will imply:

$$|f(\tfrac{a}{b})| = |\tfrac{a}{b} - \alpha| \cdot |g(\tfrac{a}{b})|$$

$$< \frac{1}{b^D} \cdot K = \frac{K}{b^D}.$$

Lower bound: $f(x) = c_0 x^d + \cdots c_d$

$$f(\tfrac{a}{b}) = c_0 (\tfrac{a}{b})^d + \cdots c_{d-1}(\tfrac{a}{b}) + c_d$$

$$= \frac{c_0 a^d + c_1 a^{d-1} b + \cdots c_{d-1} a b^{d-1} + c_d b^d}{b^d}$$

We can assume $f(\frac{a}{b}) \neq 0$ (since there are at most d- solutions for $f(x)=0$.)

Then $|c_0 a^d + \cdots c_d b^d| \geq 1$

$$\Rightarrow |f(\tfrac{a}{b})| \geq \frac{1}{b^d}.$$

Combine two results $\frac{1}{b^d} \leq |f(\frac{a}{b})| < \frac{K}{b^D}.$

This implies: $b^{D-d} < K$.

$D > d \Rightarrow$ There are only <u>finitely many</u> choice of b.

For each b, there are only finitely many a satisfy

$$\left|\frac{a}{b} - \alpha\right| < 1.$$

Therefore, there are only finitely many $\frac{a}{b}$ such

that $\left|\frac{a}{b} - \alpha\right| < \frac{1}{b^D}$. $\square$

Lemma: Let β be the Liouville's number:

$$\beta = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

Then for any $D \geq 1$, we can find infinitely many

rational numbers $\frac{a}{b}$ such that:

$$\left|\frac{a}{b} - \beta\right| < \frac{1}{b^D}.$$

Proof: Set $\beta_N = \sum_{n=1}^{N} \frac{1}{10^{n!}} = \frac{1}{10} + \frac{1}{10^2} + \cdots \frac{1}{10^{N!}}$

Obviously, $\beta_1 < \beta_2 < \beta_3 < \beta_4 \cdots\cdots < \beta$

$$\Rightarrow \quad |\beta_N - \beta| = \beta - \beta_N.$$

- Set $\beta_N = \dfrac{a_N}{b_N} \Rightarrow b_N = 10^{N!}$

- $$\beta - \beta_N = \frac{1}{10^{(N+1)!}} + \frac{1}{10^{(N+2)!}} + \cdots$$

$$< \frac{2}{10^{(N+1)!}}.$$

- $$\frac{2}{10^{(N+1)!}} = \frac{2}{\left(10^{N!}\right)^{N+1}} = \frac{2}{b_N^{N+1}} < \frac{1}{b_N^N}$$

This means: $\left|\beta_N - \beta\right| = \left|\dfrac{a_N}{b_N} - \beta\right| < \dfrac{1}{b_N^N}$.

Now: let $D > 0$ and we can choose $N > D$.

Then for any β_M with $M \geq N$

$$\left|\beta_M - \beta\right| = \left|\frac{a_M}{b_M} - \beta\right| < \frac{1}{b_M^M} < \frac{1}{b_M^D}.$$

This means, there are infinitely many rational solutions

for $\left|\dfrac{a}{b} - \beta\right| < \dfrac{1}{b^D}$.

Theorem (Transcendence of β)

Liouville's number $\beta = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is transcendental.

Proof: Suppose not. Then β is algebraic and we can find $f(x) \in \mathbb{Z}[x]$ such that $f(\beta) = 0$.

Let $d = \deg . f(x)$. Then by Liouville's inequality, there are finitely many rational solutions $\frac{a}{b}$ such that $\left| \frac{a}{b} - \beta \right| < \frac{1}{b^{d+1}}$

However, the lemma is claiming that there are infinitely many solutions such that

$$\left| \frac{a}{b} - \beta \right| < \frac{1}{b^{d+1}}.$$

A contradiction! $\square$.