(Chapter 12 P83-84)

Observation: Let $n \geqslant 2$ be an integer. Then we can always find a prime q such that $q \mid n$.

Two cases: (1) n prime $q=n$

(2) n not prime $n = q_1 \cdots q_r$ with each being prime.

Theorem: There are infinitely many prime numbers.

Euclid's proof: (Proof by contradiction)

Assume that there are finitely many primes.

Then we can list all the primes $P_1, P_2, \cdots P_n$.

We look at

$$A = P_1 P_2 \cdots P_n + 1.$$

Let q be a prime such that $q \mid A$.

Then q should be one of $P_1, \cdots P_n$. For example $q = P_1$

However, $\gcd(P_1, A) = 1 \left(= \gcd(q, A)\right)$ since

$$A - (P_2 \cdot P_3 \cdots P_n) \cdot P_1 = 1.$$

This gives ① $q \mid A$.

② $\gcd(q, A) = 1$

This can never happen at the same time.

This means: we get a contradiction:

This implies: our assumption is wrong!

Therefore, there are infinitely many primes! $\square$

Euler's proof: He looked at

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=\sum_{p \text{ prime}}\frac{1}{p} \quad \text{infinite series.}$$

He showed $\displaystyle\sum_{p \text{ prime}}\frac{1}{p}=\infty$.

Therefore, there are infinitely many primes.

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(Chapter 13.)

We proved, there are infinitely many primes.

We want to describe the number of primes in a more precise way.

Therefore, we will introduce the counting function.

Before that, we look at one easier counting function.

For $x > 0$, we define:

$$E(x) := \# \{ \text{even numbers } n \text{ with } 1 \leq n \leq x \}$$

Example:

$$E(1) = \# \{ \text{even numbers } n \text{ with } 1 \leq n \leq 1 \} = 0.$$

$$E(3) = \# \{ \text{even numbers } n \text{ with } 1 \leq n \leq 3 \}$$

$$= \# \{2\} = 1$$

$$E(4) = \# \{ \text{even numbers } n \text{ with } 1 \leq n \leq 4 \}$$

$$= \# \{2, 4\} = 2$$

...

$$E(51) = \# \{ \text{even numbers } n \text{ with } 1 \leq n \leq 51 \}$$

$$= \# \{ 2, 4, 6, 8, \cdots 48, 50 \} = 25$$

$$E(100) = \#\{\text{even numbers } n \text{ with } 1 \le n \le 100\}$$
$$= \#\{2, 4, 6, \cdots 98, 100\} = 50.$$

We look at $\frac{E(x)}{x}$.

$$\frac{E(1)}{1} = 0, \quad \frac{E(3)}{3} = \frac{1}{3}, \quad \frac{E(4)}{4} = \frac{1}{2},$$

$$\frac{E(51)}{51} = \frac{25}{51}, \quad \frac{E(100)}{100} = \frac{50}{100}.$$

Indeed, we can show, as x becomes larger and larger, $\frac{E(x)}{x}$ is closer and closer to $\frac{1}{2}$.

Therefore, we have:

$$\lim_{x\to\infty} \frac{E(x)}{x} = \frac{1}{2}. \quad \left(\text{or } E(x) \sim \frac{x}{2}.\right)$$

Next, we consider prime counting function:

$$\pi(x) := \#\{\text{primes } p \text{ with } 1 \le p \le x\}$$

Example $\pi(10) = \#\{\text{primes } p \text{ with } 1 \le p \le 10\}$
$$= \#\{2, 3, 5, 7\} = 4.$$

Theorem 13.1 (Prime Number Theorem, PNT)

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1 \quad \left(\pi(x) \sim \frac{x}{\ln x} \right)$$

- This was conjectured by Gauss and Legendre in 1800s.
- This was proved by Hadamard and de la Vallée Poussin independently.

(calculus for complex numbers)

- In their proofs, they used methods in complex analysis to study Riemann zeta funtion.

  It's suprising that, we need calculus to study integers.
  This is one of the branches of number theory
  — analytic number theory.

- In 1948, Erdös and Selberg found an elementary proof for PNT