In this section, we study the arithmetic functions.

Definition: An <u>arithmetic function</u> is a function defined over integers, i.e. $f: \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$)

Example: (1) The trivial function $\mathbb{1}: \mathbb{N} \to \mathbb{R}$

$$\mathbb{1}(n) = 1 \quad \text{for all } n \in \mathbb{N}.$$

(2) The Euler's Phi function: $\phi: \mathbb{N} \to \mathbb{R}$

$$\phi(m) := \#\{a : 1 \le a \le m, \gcd(a, m) = 1\}$$

(3) The divisor function: $d: \mathbb{N} \to \mathbb{R}$

$$d(m) := \#\{a : a \mid m\}.$$

(4) The Möbius function: $\mu: \mathbb{N} \to \mathbb{R}$.

$$\mu(m) = \begin{cases} (-1)^r & \text{if } m = p_1 p_2 \cdots p_r \text{ with } p_i \text{ distinct.} \\ 0 & \text{otherwise.} \end{cases}$$

Definition: An arithmetic function: f is <u>multiplicative</u> if $f(mn) = f(m) f(n)$ when $\gcd(m, n) = 1$.

An arithmetic function f is <u>completely multiplicative</u> if $f(mn) = f(m) f(n)$ for all m, n.

Remark: f completely multiplicative $\Rightarrow$ multiplicative.

In fact: (1) $\mathbb{1}(m)$ is completely multiplicative.

(2) $\phi(m)$ is multiplicative $\rightsquigarrow$ will show later.

but not completely multiplicative.

(counter) example: $\phi(4)=2$ $\phi(2)=1$

$$\phi(4)=\phi(2\cdot 2)\neq \phi(2)\cdot\phi(2)$$

(3) $d(m)$ is multiplicative

but not completely multiplicative.

(counter) example: $d(4)=3$ $d(2)=2$

$$d(4)=d(2\cdot 2)\neq d(2)\cdot d(2)$$

(4). $\mu(m)$ is compliative

but not completely multiplicative.

(counter) example: $4=2\cdot 2=2^2$

$\mu(4)=0$ $\mu(2)=-1$.

$$\mu(4)=\mu(2\cdot 2)\neq \mu(2)\cdot\mu(2).$$

Notations:

sum notation: $\sum$

product notation: $\prod$

example: $\sum_{p|n} p$ means: find all primes divides n and sum them.

$$\sum_{p|10} = 2+5 = 7$$

$\prod_{p|n}\left(1-\frac{1}{p}\right)$ means: multiply all $\left(1-\frac{1}{p}\right)$ where $p|n$.

$$\prod_{p|6}\left(1-\frac{1}{p}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.$$

Question: why the multiplicative functions are important.

Ans: Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with p_i distinct.

Let f be multiplicative.

Then
$$f(m) = f(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r})$$
$$= f(p_1^{\alpha_1}) f(p_2^{\alpha_2}) \cdots f(p_r^{\alpha_r})$$
$$= \prod_{p^\alpha \| m} f(p^\alpha)$$

f is totally determined by its values at prime powers.

Moreover, if f is completely multiplicative.

$$f(m) = f(p_1)^{\alpha_1} f(p_2)^{\alpha_2} \cdots f(p_r)^{\alpha_r}$$
$$= \prod_{p^\alpha \| m} f(p)^\alpha$$

f is totally determined by its values at primes.

First example: Euler's Phi funtion.

Theorem (11.1 Euler's Phi funtion formula)

(a) If p is a prime and $k \geq 1$, then

$$\phi(p^k) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right)$$

(b) If $\gcd(m,n) = 1$, then

$$\phi(mn) = \phi(m)\phi(n).$$

(c) For $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$

$$\phi(m) = \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right) \cdots \left(p_r^{\alpha_r} - p_r^{\alpha_r - 1}\right)$$

$$= m \cdot \prod_{p \mid m} \left(1 - \frac{1}{p}\right)$$

Proof of (c): By (a), (b)

$$\phi(m) = \phi(p_1^{\alpha_1})\, \phi(p_2^{\alpha_2}) \cdots \phi(p_r^{\alpha_r})$$

$$= \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right) \cdots \left(p_r^{\alpha_r} - p_r^{\alpha_r - 1}\right)$$

$$= p_1^{\alpha_1}\left(1 - \frac{1}{p_1}\right) p_2^{\alpha_2}\left(1 - \frac{1}{p_2}\right) \cdots p_r^{\alpha_r}\left(1 - \frac{1}{p_r}\right)$$

$$= m \prod_{p \mid m} \left(1 - \frac{1}{p}\right)$$