In this lecture, we study: let p be an odd prime,

$$\left(\frac{2}{p}\right) = ?$$

By Euler's cretetion, we have:

$$2^{\frac{p-1}{2}} \equiv \left(\frac{2}{p}\right) \pmod{p}$$

Therefore, we just need to investigate $2^{\frac{p-1}{2}} \pmod{p}$.

We look at two examples and then try to generalize it.

Example I: $p = 13$.   Set $P = \frac{p-1}{2} = 6$.

We look at all even integers between 0 and 13.

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12. \qquad \left(6 \text{ numbers}\right)$$

- $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 = 2(1) \cdot 2(2) \cdot 2(3) \cdot 2(4) \cdot 2(5) \cdot 2(6)$

$$= 2^6 \cdot (1)(2)(3)(4)(5)(6)$$

$$= 2^6 \cdot 6!$$

- $2 \cdot 4 \cdot \underset{\triangle}{6} \cdot 8 \cdot 10 \cdot 12 \equiv 2 \cdot 4 \cdot 6 \; (8-13)(10-13)(12-13) \pmod{13}$

$$\equiv 2 \cdot 4 \cdot 6 \cdot (-5)(-3)(-1) \pmod{13}$$

$$\equiv (-1)^3 \cdot 6!$$

$$\Rightarrow \quad 2^6 \cdot 6! \equiv (-1)^3 \cdot 6! \pmod{13}$$

$$\Rightarrow \quad 2^6 \equiv -1 \pmod{13} \Rightarrow \quad \left(\frac{2}{13}\right) = -1$$

Example II. $P = 17$ $\frac{P}{} = \frac{17-1}{2} = 8$

We look at all even integers between 0 and 17.

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad (8 \text{ numbers})$$

- $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 = 2(1) \cdot 2(2) \cdot 2(3) \cdot 2(4) \cdot 2(5) \cdot 2(6) \cdot 2(7) \cdot 2(8)$

$$= 2^8 \cdot 8! \qquad \qquad \pmod{17}$$

- $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \equiv 2 \cdot 4 \cdot 6 \cdot 8 \cdot (10-17)(12-17)(14-17)(16-17)$

$$\equiv 2 \cdot 4 \cdot 6 \cdot 8 \cdot (-7)(-5)(-3)(-1) \pmod{17}$$

$$\equiv (-1)^4 \cdot 8! \pmod{17}$$

$$\Rightarrow \quad 2^8 \cdot 8! \equiv (-1)^4 \cdot 8! \pmod{17}$$

$$\Rightarrow \quad 2^8 \equiv 1 \pmod{17} \qquad \left(\frac{2}{17}\right) = 1.$$

We consider a general p, set $P = \frac{p-1}{2}$.

- $2 \cdot 4 \cdot 6 \cdots (p-1) = 2(1) \cdot 2(2) \cdot 2(3) \cdots 2 \cdot \left(\frac{p-1}{2}\right)$

$$= 2^{\frac{p-1}{2}} \cdot \left(\frac{p-1}{2}\right)!$$

- $2 \cdot 4 \cdot 6 \cdots (p-1) = 2 \cdot 4 \cdot 6 \cdots \frac{p-1}{2} \cdot \frac{p+3}{2} \; \frac{p+7}{2} \cdots (p-1)$

$$\equiv 2 \cdot 4 \cdot 6 \cdots \frac{p-1}{2} \cdot \left(\frac{p+3}{2} - p\right)\left(\frac{p+7}{2} - p\right) \cdots (p+1-p) \pmod{p}$$

$$\equiv 2 \cdot 4 \cdot 6 \cdots \frac{p-1}{2} \cdot \left(-\frac{p-3}{2}\right)\left(-\frac{p-7}{2}\right) \cdots (-1) \pmod{p}$$

$$\equiv (-1)^{*} \cdot \left(\frac{p-1}{2}\right)! \pmod{p}$$

$(*) =$ number of integers in the list

$$2, 4, 6, 8, \cdots p-1$$

that are larger than $\frac{p-1}{2}$.

$\Rightarrow \quad 2^{\frac{p-1}{2}} \cdot \left(\frac{p-1}{2}\right)! \equiv (-1)^{*} \cdot \left(\frac{p-1}{2}\right)! \pmod{p}$

$\Rightarrow \quad 2^{\frac{p-1}{2}} \equiv (-1)^{*} \pmod{p} \quad \Rightarrow \quad \left(\frac{2}{p}\right) = (-1)^{*}$

Here we only considered the case when P is even. We can do this in a similar way when P is odd.

Theorem 21.4 (Quadratic Reciprocity, Part II)

Let p be an odd prime.

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod{8} \\ -1 & \text{if } p \equiv 3, 5 \pmod{8} \end{cases}$$

Proof: We need to consider 4 cases. Here we only do

$$p \equiv 1 \pmod 8 \quad \text{and} \quad p \equiv 5 \pmod 8$$

Case: $p \equiv 1 \pmod 8$ $\qquad p = 8k + 1.$ $\qquad P = \frac{p-1}{2} = 4k.$

$$2, 4, 6, \cdots 4k, 4k+2, \cdots 8k \qquad (4k \text{ numbers})$$

We need to count how many numbers are $> P = 4k$

That is, $\quad 4k+2, 4k+4, \cdots 8k.$

There are $2k$ numbers in total and hence

$$2^{\frac{p-1}{2}} \equiv (-1)^{2k} \pmod p$$
$$\equiv 1 \pmod p$$
$$\Rightarrow \quad \left(\frac{2}{p}\right) = 1$$

Case: $p \equiv 7 \pmod 8$ $p = 8k + 7$. $P = \frac{p-1}{2} = 4k + 3$

$$2, 4, 6, \cdots 4k, 4k+2, 4k+4, \cdots 8k+6$$

We need to count how many numbers are $> P = 4k + 3$

That is, $4k+4, \cdots 8k+6$

There are $2k+2$ numbers in total and hence

$$2^{\frac{p-1}{2}} \equiv (-1)^{2k+2} \pmod p \equiv 1 \pmod p$$

$$\Rightarrow \left(\frac{2}{p}\right) = 1$$

Case: $p \equiv 3 \pmod 8$ $p = 8k + 3$. $P = \frac{p-1}{2} = 4k + 1$

$$2, 4, 6, \cdots 4k, 4k+2, \cdots 8k+2$$

We need to count how many numbers are $> P = 4k + 1$

That is, $4k+2, 4k+4, \cdots 8k+2$

There are $2k+1$ numbers in total and hence

$$2^{\frac{p-1}{2}} \equiv (-1)^{2k+1} \pmod p \equiv -1 \pmod p$$

$$\Rightarrow \left(\frac{2}{p}\right) = -1$$

Case: $p \equiv 5 \pmod 8$ $p = 8k+5$ $\frac{p}{2} = \frac{p-1}{2} = 4k+2$.

We look at

$$2, 4, 6, \cdots 4k+2, \underbrace{4k+4, 4k+6, \cdots 8k+4}_{2k+1 \text{ numbers.}}$$

Therefore, $2^{\frac{p-1}{2}} \equiv (-1)^{2k+1} \pmod p \equiv -1 \pmod p$

$$\Rightarrow \left(\frac{2}{p}\right) = -1$$

Remark: In the proof of the theorem, the key observation is:

$$2^{\frac{p-1}{2}} \equiv (-1)^{\#} \pmod p$$

where $\#$ = number of integers in the list

$$2, 4, 6, 8, \cdots p-1$$

that are larger than $\frac{p-1}{2}$.

Here is another way to interpret $2, 4, 6, 8, \cdots p-1$:

$$2 \cdot 1, \; 2 \cdot 2, \; 2 \cdot 3, \; \cdots 2 \cdot \frac{p-1}{2}.$$

Later, when we consider $\left(\frac{a}{p}\right)$, we will look at:

$$a \cdot 1, \; a \cdot 2, \; a \cdot 3, \; \cdots a \cdot \frac{p-1}{2}.$$