

Denote by $i = \sqrt{-1}$. The complex numbers are defined by

$$\mathbb{C} = \{z = a + bi \mid a, b \in \mathbb{R}\}$$

Let $a + bi, c + di \in \mathbb{C}$, we define:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Theorem: $(\mathbb{C}, +, \cdot)$ is a field.

Before the proof, we introduce the conjugate of a complex

number: let $z = a + bi$ with $a, b \in \mathbb{R}$, then the

conjugate of z , denoted by \bar{z} , is $\bar{z} = a - bi$

Observation:

$$\begin{aligned} z \cdot \bar{z} &= (a + bi) \cdot (a - bi) && \geq 0. \\ &= (a^2 + b^2) + (a(-b) + ab)i = a^2 + b^2 \in \mathbb{R}. \end{aligned}$$

The norm of z , denoted by $|z|$, is:

$$|z| = \sqrt{a^2 + b^2} = (z \cdot \bar{z})^{\frac{1}{2}}.$$

If $z \neq 0$, $a, b \neq 0$, then $|z| > 0$.

This also shows: , for $z \neq 0$, $z \cdot \frac{\bar{z}}{|z|^2} = \frac{z \cdot \bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$.

Proof of Theorem:

(I) For $+$:

(a) (Associative) let $x_1 + y_1 i$, $x_2 + y_2 i$, $x_3 + y_3 i \in \mathbb{C}$,

$$\begin{aligned} & x_1 + y_1 i + \left((x_2 + y_2 i) + (x_3 + y_3 i) \right) \\ &= x_1 + y_1 i + (x_2 + x_3) + (y_2 + y_3) i \end{aligned}$$

$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) i$$

$$\left((x_1 + y_1 i) + (x_2 + y_2 i) \right) + (x_3 + y_3 i)$$

$$= (x_1 + x_2) + (y_1 + y_2) i + x_3 + y_3 i$$

$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) i. \quad \checkmark$$

(b) (Commutative) $x + y i$, $a + b i \in \mathbb{C}$

$$(x + y i) + (a + b i) = (x + a) + (y + b) i$$

$$= (a + x) + (b + y) i$$

$$= (a + b i) + (x + y i) \quad \checkmark$$

(c) (Additive identity) : we can show:

$$a + b i + 0 = (a + 0) + b i = a + b i \quad \checkmark$$

(d) (Additive inverse) : for $a+bi \in \mathbb{C}$,

$$a+bi + (-a-bi) = (a-a) + (b-b)i = 0+0i=0$$

(2) For •

(a) (Associative)

$$(a+bi)((c+di)(x+yi))$$

$$= (a+bi)(cx-dy + (cy+dx)i)$$

$$= a(cx-dy) - b(cy+dx)$$

$$+ b(cx-dy)i + a(cy+dx)i$$

$$= (acx - ady - bcy - bdx)$$

$$+ (bcx - bdy + acy + adx)i$$

$$((a+bi)(c+di))(x+yi)$$

$$= ((ac-bd) + (ced+bc)i)(x+yi)$$

$$= (ac-bd)x - y(ad+bc)$$

$$+ (ac-bd)y i + (ad+bc)x i$$

$$= (acx - bdx - yad - ybc)$$

$$+ (acy - bdy + adx + bcx)i \quad \checkmark$$

(b) Commutative: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$$(c+di)(a+bi) = ca-db + (da+cb)i \\ = ac-bd + (ad+bc)i \quad \checkmark$$

(c) (Multiplicative identity) $(a+bi) \cdot 1 = (a+bi)(1+0 \cdot i)$

$$= (a-b \cdot 0) + (a \cdot 0 + b \cdot 1)i \\ = a+bi \quad \checkmark$$

(d) (Multiplicative inverse) $a+bi \neq 0 \Rightarrow a, b \neq 0$

$$(a+bi) \cdot \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i \right) \\ = \left(a \cdot \frac{a}{a^2+b^2} - b \left(-\frac{b}{a^2+b^2} \right) \right) + \left(a \cdot \left(\frac{-b}{a^2+b^2} \right) + b \left(\frac{a}{a^2+b^2} \right) \right) i \\ = \frac{a^2+b^2}{a^2+b^2} = 1$$

(3) $(a+bi)(x_1+y_1i + x_2+y_2i)$

$$= (a+bi)((x_1+x_2) + (y_1+y_2)i) \\ = a(x_1+x_2) - b(y_1+y_2) + (a(y_1+y_2) + b(x_1+x_2))i \\ (a+bi)(x_1+y_1i) + (a+bi)(x_2+y_2i) \\ = (ax_1 - by_1) + (ay_1 + bx_1)i + (ax_2 - by_2) + (ay_2 + bx_2)i$$

$$= (ax_1 + ax_2 - by_1 - by_2) + (ay_1 + ay_2 + bx_1 + bx_2) \quad \checkmark$$