Recall: $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

integers, rational numbers, real numbers, complex numbers.

Let α be a complex number.

Definition: α is an <u>algebraic integer</u> if we can find a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$

with $a_{n-1}, \cdots, a_1, a_0 \in \mathbb{Z}$

such that $f(\alpha) = 0$.

Definition: α is an <u>algebraic number</u> if we can find a polynomial $g(x) = a_n x^n + \cdots a_1 x + a_0$

with $a_n, a_{n-1}, \cdots a_1, a_0 \in \mathbb{Q}$

such that $g(\alpha) = 0$.

Remark: (1) If α is an algebraic integer, then α is an algebraic number. This is because: $\mathbb{Z} \subseteq \mathbb{Q}$.

(2) Let α be an algebraic number. Then we can find $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$ $a_n, \cdots a_0 \in \mathbb{Q}$.

such that $g(\alpha) = 0$.

Sine $a_n, \dots a_0$ are rational numbers, we can write:

$$a_n = \frac{r_n}{s_n}, \quad a_{n-1} = \frac{r_{n-1}}{s_{n-1}}, \quad \cdots \quad a_1 = \frac{r_1}{s_1}, \quad a_0 = \frac{r_0}{s_0}$$

with $r_n, \dots r_0, s_n, \cdots s_0 \in \mathbb{Z}$.

Then we multiply $g(x)=0$ by $s_n s_{n-1} \cdots s_0$

we get a new polynomial

$$\tilde{g}(x) = \tilde{a}_n x^n + \cdots \tilde{a}_1 x + \tilde{a}_0 \qquad \tilde{a}_n, \dots \tilde{a}_0 \in \mathbb{Z}$$

such that $\tilde{g}(\alpha) = 0$.

This shows that: if α is an algebraic number, then we can find a polynomial $\tilde{g}(x)$ with coefficients in $\mathbb{Z}$ such that $\tilde{g}(\alpha)=0$.

Therefore, the difference between algebraic integers and algebraic numbers is:

$f(\alpha)=0$ for $f(x) = 1 \cdot x^n + a_{n-1}x^{n-1} + \dots a_0$ $\rightsquigarrow$ algebraic integers

$g(\alpha)=0$ for $g(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_0$. $\rightsquigarrow$ algebraic numbers

leading coefficient.

Definition: A polynomial is <u>monic</u> if the leading coefficient is 1.

Some notations: let A be a set.

$$A[x] = \{ \text{polynomials with coefficients in } A \}$$

Definition': α is an <u>algebraic integer</u> if we can find a monic $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$

<u>Definition'</u>: α is an <u>algebraic number</u> if we can find $g(x) \in \mathbb{Z}[x]$ such that $g(\alpha) = 0$.

Example: (1) $i = \sqrt{-1}$ is an algebraic integer (and hence an algebraic number)

This is because: $i^2 = -1$ and hence i is a solution for $f(x) = x^2 + 1$.

(2) $\sqrt{2} + \sqrt{3}$ is an algebraic integer (and hence an algebraic number).

Proof: Set $x = \sqrt{2} + \sqrt{3}$ Then $x - \sqrt{2} = \sqrt{3}$

$\Rightarrow (x - \sqrt{2})^2 = (\sqrt{3})^2 \Rightarrow x^2 - 2\sqrt{2}x + 2 = 3$

$\Rightarrow x^2 - 1 = 2\sqrt{2}x$

$\Rightarrow (x^2 - 1) = (2\sqrt{2}x) \Rightarrow x^4 - 2x^2 + 1 = 8x^2$

$\Rightarrow \quad x^4 - 10x^2 + 1 = 0.$

Therefore, $\sqrt{2}+\sqrt{3}$ is a solution for $x^4 - 10x^2 + 1 = 0$ and hence it is an algebraic integer.

(3) $\frac{2}{3}$ is an algebraic number but not an algebraic integer!

$g(x) = 3x - 2$. $\quad g(\frac{2}{3}) = 0$. $\Rightarrow$ algebraic number

Fact: Let $\alpha \in \mathbb{Q}$ be an algebraic integer. Then $\alpha \in \mathbb{Z}$.

Proposition: Let α be an algebraic number. Then we can find an integer $d \in \mathbb{Z}$ such that $d\alpha$ is an algebraic integer

Proof: α is an algebraic number. Then we can find $f(x) \in \mathbb{Z}[x]$ such that

$$f(\alpha) = a_n\alpha^n + a_{n-1}x^{n-1} + \dots + a_1\alpha + a_0 = 0. \quad a_n \in \mathbb{Z}.$$

Then we multiply the whole equation by a_n^{n-1}:

$$(a_n\alpha)^n + a_{n-1}\cdot(a_n\alpha)^{n-1} + \dots a_1 a_n^{n-2}(a_n\alpha) + a_0 a_n^{n-1} = 0$$

We set $g(x) = x^n + a_{n-1}x^{n-1} + \dots a_1 a_n^{n-2} x + a_0 a_n^{n-1}$

and $g(a_n\alpha) = 0 \Rightarrow a_n\alpha$ is an algebraic integer.

$\square$

Definition: Let $\alpha \in \mathbb{C}$. α is a transcendental number if α is <u>not</u> an algebraic number

In other words, we can <u>never</u> find $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$.

Example: e, π are transcendental numbers.

However, this is <u>very difficult</u> to prove!

Fact: There are <u>more</u> transcendental numbers than algebraic numbers.