

The Pigeonhole Principle.

Suppose that there are $n+1$ pigeons but n pigeonholes, then at least one pigeonhole contains more than one pigeon.

Theorem 33.1 Dirichlet's Diophantine Approximation Theorem.

Let D be a positive integer that is not a square.

Then there are infinitely many pairs of positive integers

(x, y) such that

$$|x - y\sqrt{D}| < \frac{1}{y}.$$

Remark: (1) When D is not a square, \sqrt{D} is irrational.

This is the only place where we use that D is not a square. Indeed, this theorem can be generalized to any irrational numbers

(2) Here is another way to write

$$|x - y\sqrt{D}| < \frac{1}{y}$$

that is: $\left| \frac{x}{y} - \sqrt{D} \right| < \frac{1}{y^2}$

This means: for \sqrt{D} (irrational), we can always find rational numbers $\frac{x}{y}$ that is close to \sqrt{D} .
That's why we say this is an "approximation."

Proof of Theorem: Let Y be a large integer.

We investigate:

$Y+1$
numbers.

$$\begin{aligned} 0 \cdot \sqrt{D} &= N_0 + F_0 & N_0=0, F_0=0 \\ 1 \cdot \sqrt{D} &= N_1 + F_1 & N_1 \text{ integer, } 0 \leq F_1 < 1 \\ & & (N_1 = \lfloor \sqrt{D} \rfloor) \\ 2 \cdot \sqrt{D} &= N_2 + F_2, & N_2 \text{ integer, } 0 \leq F_2 < 1 \\ & & (N_2 = \lfloor 2\sqrt{D} \rfloor) \\ & \dots & \\ Y \cdot \sqrt{D} &= N_Y + F_Y & N_Y \text{ integer, } 0 \leq F_Y < 1 \\ & & (N_Y = \lfloor Y\sqrt{D} \rfloor) \end{aligned}$$

$\Rightarrow F_0, F_1, \dots, F_Y$ are $Y+1$ integers (pigeons) in $[0, 1)$.

We divide $[0, 1)$ into several smaller intervals (pigeonholes)

$$[0, \frac{1}{Y}), [\frac{1}{Y}, \frac{2}{Y}), [\frac{2}{Y}, \frac{3}{Y}), \dots, [\frac{Y-1}{Y}, 1)$$

F_0, F_1, \dots, F_Y ($Y+1$ pigeons) must fall into.

Y intervals (Y pigeonholes)

By Pigeonhole principle, we can find $0 \leq m < n \leq Y$
 such that F_m and F_n is the same interval
 (same pigeon hole)

This shows: $|F_m - F_n| < \frac{1}{Y}$

since each interval is of length $\frac{1}{Y}$.

$$m\sqrt{D} = N_m + F_m \Rightarrow F_m = m\sqrt{D} - N_m \quad \text{and} \\ n\sqrt{D} = N_n + F_n \Rightarrow F_n = n\sqrt{D} - N_n \quad N_m < N_n$$

$$\Rightarrow |F_m - F_n| = |(N_n - N_m) - (n-m)\sqrt{D}| < \frac{1}{Y}$$

Then set $x = N_n - N_m$ and $y = n - m$

Since $0 \leq m < n \leq Y \Rightarrow y = n - m \leq Y \Rightarrow y \leq Y \Rightarrow \frac{1}{Y} < \frac{1}{y}$.

This becomes $|x - y\sqrt{D}| < \frac{1}{y}$.

We next show: there are infinitely many pairs. Suppose not, we can find $(x_1, y_1) \dots (x_n, y_n)$ with

$$|x_i - y_i \sqrt{D}| < \frac{1}{y_i} \quad i=1, 2, \dots, n$$

Then we can find a such that:

$$|x_1 - y_1 \sqrt{D}|, |x_2 - y_2 \sqrt{D}|, \dots, |x_n - y_n \sqrt{D}| > a.$$

Then we can find an integer γ' such that

$$\frac{1}{\gamma'} > a$$

We run the argument above before, and we can

find (x_0, y_0)

$$|x_0 - y_0 \sqrt{D}| < \frac{1}{y_0} \quad \text{and} \quad |x_0 - y_0 \sqrt{D}| < \frac{1}{\gamma'} < a$$

The last " $<$ " guarantees that (x_0, y_0) is a new pair.

A contradiction!

□

Next, we replace \sqrt{D} by any irrational integers:

Theorem 33.2 (Dirichlet's Diophantine Approximation Theorem)

Let $\alpha > 0$ be an irrational number. Then there are infinitely many pairs of positive integers (x, y) such that

$$|x - y\alpha| < \frac{1}{y}.$$